MATH 8102 Fall 2026 — Scribe: Elena Bou Aoun

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Lecture 6: Putinar's Positivstellensatz (Theobald §6.7)

Lemma 1 (Lemma 6.43). *If $S(g_1, \ldots, g_m)$ is compact and $g_1, \ldots, g_m$ have Putinar's property, then, for any $g_{m+1} \in \mathbb{R}[x]$, the polynomials*

$$g_1, \ldots, g_m, g_{m+1}$$

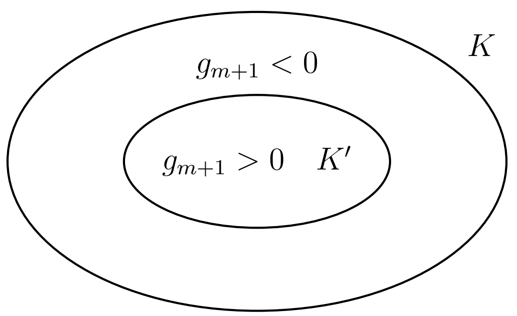
have Putinar's property.

Proof. Set

$$K = S(g_1, \ldots, g_m)$$

and

$$K' = S(g_1, \ldots, g_m, g_{m+1}).$$



Suppose $f \in \mathbb{R}[\mathbf{x}]$ and $f > 0 \quad$ on K'. To show that

$$f \in QM(g_1, \ldots, g_m, g_{m+1}),$$

it is enough to show that

$$f - \sigma_{m+1} g_{m+1} > 0 \quad \text{on } K$$

for some s.o.s. σ_{m+1}.

Note that If $f > 0$ on K, we can take $\sigma_{m+1} = 0$.

Set

$$D = K \setminus K',$$

and suppose there exists $y \in D$ such that

$$f(y) \leq 0.$$

Define

$$M = \max_{x \in D} \left\{ \frac{f(x)}{g_{m+1}(x)} \right\} + 1.$$

Note that D has compact closure, and

$$\frac{f}{g_{m+1}} \longrightarrow -\infty$$

as

$$x \longrightarrow \overline{D} \setminus D,$$

since f is positive near the boundary. Hence, the maximum exists. Since

$$f(y) \leq 0 \qquad \text{and} \qquad g_{m+1}(y) < 0,$$

we have

$$M > 0.$$

Now define

$$\widetilde{\sigma}_{m+1}(x) = \begin{cases} \min\left(M, \dfrac{f(x)}{2g_{m+1}(x)}\right), & \text{if } g_{m+1}(x) > 0, \\ M, & \text{if } g_{m+1}(x) \leq 0. \end{cases}$$

Claim: $\widetilde{\sigma}_{m+1}$ is continuous, and $\widetilde{\sigma}_{m+1} > 0 \quad \text{on } K$.

Moreover, $f - \widetilde{\sigma}_{m+1} g_{m+1} > 0 \quad \text{on } K$, and continuous. Indeed, on K',

$$f - \widetilde{\sigma}_{m+1} g_{m+1} \geq f - \frac{f}{2g_{m+1}} g_{m+1} = \frac{f}{2} > 0.$$

On D,

$$f - M g_{m+1} \geq f - \left(1 + \frac{f}{g_{m+1}}\right) g_{m+1} = -g_{m+1} > 0.$$

Recall that Any continuous function on a compact subset of $\mathbb{R}^n$ can be uniformly approximated by a polynomial function by the Stone–Weierstrass theorem.

Use Stone–Weierstrass to uniformly approximate

$$\sqrt{\widetilde{\sigma}_{m+1}(x)}$$

by a polynomial $r(\mathbf{x})$ such that

$$f - r^2 g_{m+1} > 0 \quad \text{on } K.$$

□

Example 1. *Claim: For any $N > 0$, the polynomial*

$$g = N - \sum_{i=1}^{n} x_i^2$$

has the Putinar property.

Without loss of generality, take $N = 1$. Then the zero set of g is the unit sphere $S^{n-1} \subseteq \mathbb{R}^n$. Consider the point

$$p = (1, 0, \ldots, 0) \in S^{n-1}.$$

The tangent space $T_p S^{n-1}$ is the zero set of the linear polynomial $h_1 = 1 - x_1$.

Observe that

$$1 - x_1 = \frac{1}{2}\left((x_1 - 1)^2 + x_2^2 + \cdots + x_n^2 + g(x)\right), \tag{1}$$

hence

$$1 - x_1 \in QM(g).$$

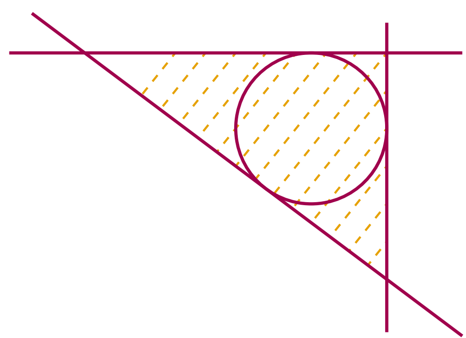
Thus, for every $p \in S^{n-1}$, the tangent space $T_p S^{n-1}$ has a linear equation belonging to $QM(g)$. Therefore, there exist tangent hyperplanes

$$h_1, \ldots, h_{n+1} \in QM(g)$$

such that

$$\Delta = S(h_1, \ldots, h_{n+1})$$

is a simplex circumscribing $S(g)$.



Suppose f is a polynomial, $f > 0$ on Δ. By Handelman's theorem,

$$f = \sum_{\beta} c_\beta h^\beta, \qquad c_\beta > 0.$$

For a monomial $h^\beta = h_1 h^{\beta'}$, *use* (1) *to write*

$$h^\beta = \frac{1}{2}\big((x_1-1)^2 + x_2^2 + \cdots + x_n^2\big)h^{\beta'} + \frac{1}{2}gh^{\beta'}.$$

Iterating with the analogous identities for the other h_i *expresses* f *as* $\sum_k \sigma_k g^k$, *where each* σ_k *is a sum of squares, hence* $f \in QM(g)$.

Finally, suppose $f > 0$ *on* $S(g)$. *As in the proof of Lemma 1, there exists a polynomial* r *with*

$$f - r^2 g > 0 \quad \textit{on } \Delta.$$

Thus $f - r^2 g \in QM(g)$, *showing that* $f \in QM(g)$.