

Lecture 14 Rolle's Theorem and Descartes Law of Signs

Corollaries of Rolle's Theorem Let $f \in R(t)$, $a < b$ in R such that f has no poles in $[a, b]$. (a) Mean Value Theorem: there exists $z \in (a, b)$ such that $f'(z) = \frac{f(b)-f(a)}{b-a}$
 (b) If $f'(x) \geq 0$ for all $x \in (a, b)$, then f is monotone increasing (non-decreasing)

Proof. (Sketch) (a) Apply Rolle's theorem to

$$g(t) = f(t) - \left[\left(\frac{f(b) - f(a)}{b - a} \right) (t - a) + f(a) \right]$$

(b) Follows from (a): $x_1, x_2 \in (a, b)$ then M.V.T tells you there is a $z \in (x_1, x_2)$ such that

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(z) \geq 0$$

which implies $f(x_2) \geq f(x_1)$ □

Descartes Law of Signs Let $f = \sum_{i=0}^n a_i t^i \in R[t] \setminus 0$. The number of strictly positive roots of f , counted with multiplicity, is at most $\text{Var}(a_0, \dots, a_n) = \#$ of sign changes (delete zeroes).

Notation: $N_+(f) = \#$ of strictly positive roots with multiplicity

Proof. Throughout this proof, count roots with multiplicities.

For $n = 1$, easy. $f = a_1 t + a_0$, f has at most 1 positive root. Must have $a_1, a_0 < 0$ if f has any positive root $\implies \text{Var}(a_0, a_1) = 1$ (if non-constant).

Fix $n > 1$, assume result for all polynomials of degree less than n . WLOG, $f(0) = a > 0$ (if < 0 , replace f with $-f$, and if zero then delete zeroes and induct on $\frac{f}{t^{\text{ord}_0(f)}}$)

Consider the smallest $r \geq 1$ such that $a_r \neq 0$ (exists if f non-constant). By Rolle's Theorem: $N_+(f) \leq N_+(f') + 1$. If we can improve this to $N_+(f) \leq N_+(f')$, then we are done since then $N_+(f) \leq N_+(f') \leq \text{Var}(a_r, \dots, a_n) \leq \text{Var}(a_0, \dots, a_n)$. Thus, it remains to consider the case where f is such that $N_+(f) = N_+(f') + 1$. Enough to show

$$\begin{aligned} (\star) \quad \text{Var}(a_0, \dots, a_n) &= \text{Var}(a_r, \dots, a_n) + 1 \\ &\geq N_+(f') + 1 \geq N_+(f) \end{aligned}$$

by induction.

Goal: $a_r < 0$, let $m = N_+(f)$. $x_1, \dots, x_m$ positive roots of f . $y_1, \dots, y_m$ positive roots of f' , so $y_i \in (x_i, x_{i+1})$ for all i since $f(0) = a_0 > 0$. f monotone on $[0, x_1] \implies$ decreasing. □