

Lecture 13 Sign changes and Real Roots

Throughout R is a real-closed field. for $f(t) = \sum_{i=0}^n a_i b_i \in R[t]$, a root $\alpha \in R(i)$ is *real* if $\alpha \in R$. Say f is real-valued if all other roots lie in R .

Sign Changes

Let $f(t) \in R(t) \setminus 0$ such that $f(a) \in R$ for some $a \in R$. Recall: if $f(t) = (t-a)^n \frac{g(t)}{h(t)}$ with $g(t), h(t) \in R[t]$ such that $g(a)h(a) \neq 0$, then we write $\text{ord}_a(f) = n$

$f(a)$ defined $\implies \text{ord}_a(f) \geq 0$. We say f *changes sign* at a if $\text{ord}_a(f)$ is odd.

Lemma 1 (1.3.7). *If $f(t) \in R(t) \setminus 0$, $a \in R$, has $f(a) = 0$, then $f'(t)/f(t)$ (which is called the logarithmic derivative) changes sign at a (from negative to positive, i.e. $\frac{f'}{f}(a-\varepsilon) < 0$, $\frac{f'}{f}(a+\varepsilon) > 0$ for all ε sufficiently small)*

Proof. (this proof was started on 9/28 and completed on 9/30) Let $n = \text{ord}_a(f)$, $f(t) = (t-a)^n g(t)$ for $g(t) \in R(t)$, $g(a) \neq 0$. Then $f'(t) = (t-a)^{n-1}[g'(t)(t-a) + ng(t)]$.

$$\frac{f'(t)}{f(t)} = \frac{n}{t-a} + \frac{g'(t)}{g(t)} = (t-a)^{-1} \left[\frac{g'(t)(t-a) + ng(t)}{g(t)} \right].$$

Define $h(t) := (t-a) \frac{f'(t)}{f(t)}$. Choose $\varepsilon > 0$ in R such that $h(t)$ has no roots or poles in $(a-\varepsilon, a+\varepsilon)$: this is possible because $h(a) = n > 0$ so a is not a pole or root of h , and h has only finitely many zeroes and poles.

By I.V.T, h must have constant sign on $(a-\varepsilon, a+\varepsilon)$. Thus $h > 0$ on $(a-\varepsilon, a+\varepsilon)$. Hence $\frac{f'}{f}$ is negative for $t \in (a-\varepsilon, a)$ and positive for $t \in (a, a+\varepsilon)$. □

Proposition 1. *Let $f \in R(t)$, $a < b$ in R such that $f(a) = f(b) = 0$. If f has no poles or roots on (a, b) , then f' has an odd number of roots on (a, b) counted with multiplicity.*

Proof. Choose $a_1, b_1 \in R$ such that $a < a_1 < b_1 < b$ and f' has no roots on $(a, a_1] \cup [b_1, b)$. For $\varepsilon > 0$ sufficiently small, $\frac{f'}{f}(a+\varepsilon) > 0$ by the previous lemma. Then by the I.V.T $\frac{f'}{f}(a_1) > 0$.

Similarly $\frac{f'}{f}(b-\varepsilon) < 0 \xrightarrow{\text{I.V.T}} \frac{f'}{f}(b_1) < 0$. Apply I.V.T to $\frac{f'}{f}$ on $[a_1, b_1]$. By the previous lemma $\frac{f'}{f}$ has an odd number of roots counted with multiplicity on $[a_1, b_1]$. Hence f' must satisfy the same count on (a, b) . □

Theorem 1 (Rolle's). For $f(t) \in R(t)$, $f(a) = f(b)$, $a < b$ in R , f has no poles on (a, b) . There exists $z \in (a, b)$ such that $f'(z) = 0$.

Proof. WLOG assume $f \not\equiv 0$, b is the smallest root of f larger than a . Now apply previous proposition. □

(Definition 1.3.11) Given a finite sequence $c = (c_1, \dots, c_n)$, $c_i \in R$ let $\text{Var}(c) = \text{Var}(c_1, \dots, c_n)$ denote the number of sign changes after deleting all zeroes from c . (Since R is real-closed, the sign function is uniquely determined).

Ex $c = (1, -1, 0, 0, -1, 0, 1, 0, -1) \implies \text{Var}(c) = 3$.

Descartes Law of Signs Let $f = \sum_{i=0}^n a_i t^i \in R[t] \setminus 0$. The number of strictly positive roots of f , counted with multiplicity, is at most $\text{Var}(a_0, \dots, a_n)$.

Notation: $N_+(f) = \#$ of strictly positive roots with multiplicity

Easy case: Suppose $N_+(f) \leq N_+(f')$. Then $N_+(f) \leq N_+(f') \leq \text{Var}(a_0, 2a_1, \dots, na_n) \leq \text{Var}(a_1, \dots, a_n)$. In general, $N_+(f) \leq N_+(f') + 1$ by Rolle's Theorem.