

## Real Closed Fields (cont.) and IVT

**Proposition (1.2.9).** *If  $K$  is real-closed, then  $K(\sqrt{-1})$  is algebraically closed.*

*Proof.* Write  $K(i) = K(\sqrt{-1})$ . Recall from the previous lecture that in the proof of proposition 1.2.7 we showed if  $L/K$  is a proper algebraic closure of  $K$  with normal closure  $\bar{L}$ , then  $G = \text{Gal}(\bar{L}/K)$  is a 2-group which implies that any proper algebraic extension of  $K(i)$  has degree which is some power of 2. Repeating the Galois correspondence argument from the same proof shows that  $K(i)$  must have a proper algebraic extension of degree 2. Thus it suffices to show that  $K(i)$  has no such extension.

Let  $w \in K(i)$  and write  $w = u + iv$  where  $u, v \in K$ . Now if  $v = 0$  and  $u \geq 0$  then

$$\sqrt{w} = \sqrt{u} \in K \subset K(i)$$

since  $K$  contains all square roots of non-negative elements of  $K$  by Proposition 1.2.3. Similarly if  $v = 0$  and  $u < 0$ , then

$$\sqrt{w} = i\sqrt{-u} \in K(i).$$

If  $v \neq 0$ , set

$$X := \sqrt{\frac{1}{2} \left( u + \sqrt{u^2 + v^2} \right)} \in K \setminus \{0\}$$

and note that  $\sqrt{u^2 + v^2} > 0$  and also  $u + \sqrt{u^2 + v^2} > u + |u| \geq 0$  where  $|u|$  is absolute value of  $u$  with respect to the order on  $K$ . Now put

$$Z := X + \frac{iv}{2X} \in K(i).$$

Computing then

$$\begin{aligned} Z^2 &= \left( X + \frac{iv}{2X} \right)^2 \\ &= \frac{1}{2} \left( u + \sqrt{u^2 + v^2} \right) - \frac{v^2}{2 \left( u + \sqrt{u^2 + v^2} \right)} + iv \\ &= \frac{\left( u + \sqrt{u^2 + v^2} \right)^2 - v^2}{2 \left( u + \sqrt{u^2 + v^2} \right)} + iv \\ &= u + iv \\ &= w. \end{aligned}$$

Thus  $K(i)$  contains all square roots, i.e. it has no quadratic extension. □

**Exercise.** Prove if  $K$  is any real field with  $K(i)$  algebraically closed, then  $K$  is real-closed.

**Corollary.** If  $K$  is real closed, any irreducible polynomial in  $K[t]$  has the form either  $t - a$  or  $t^2 + at + b$  for some  $a, b \in K$ .

Note  $t^2 + at + b$  is irreducible if and only if  $a^2 < 4b$ .

**Remarks:**

- If  $f \in K[t]$  is monic and irreducible of degree 2, and  $K$  is real closed, then  $f(a) > 0$  for all  $a \in K$ .
- If  $(K, \leq)$  is real ordered,  $L = K(i)$  then  $\text{Aut}(L/K) = \langle z \mapsto \bar{z} \rangle$ .
- For any  $z \in K(i)$ ,  $|z| = z\bar{z}$ .
- (Cauchy-Schwartz) For any  $\vec{x}, \vec{y} \in K^n$  then

$$\left( \sum_{i=1}^n x_i \bar{y}_i \right)^2 \leq \left( \sum_{i=1}^n |x_i|^2 \right) \left( \sum_{i=1}^n |y_i|^2 \right)$$

**Definition.** The **order topology** on  $K^n$  (where  $K$  is real closed) is generated by balls

$$B_r(\vec{x}) = \{ \vec{y} \in K^n : |\vec{y} - \vec{x}| < r \}$$

where  $<$  is the order on  $K$  and

$$|\vec{x}| = \sqrt{\left( \sum_{i=1}^n |x_i|^2 \right)} \in K.$$

Note when  $K = \mathbb{R}$  this is the usual Euclidean topology.

Arithmetic operations in  $K$  define continuous functions from  $K \times K \rightarrow K$ . For example  $(x, y) \mapsto x + y$  is continuous.

### Section 1.3: Real zeros of univariate polynomials.

**Note:** Throughout this section  $R$  is a real closed field and  $R_+ = R^2$  is the unique positive cone.

**Lemma (1.3.1).** *Let  $f = t^n + a_1 t^{n-1} + \cdots + a_{n-1} t + a_n$  where  $a_1, \dots, a_n \in R(\sqrt{-1})$ . Any root  $\alpha \in R(\sqrt{-1})$  of  $f$  satisfies*

$$|\alpha| < \max\{1, |a_1| + \cdots + |a_n|\}.$$

*Proof.* If  $|\alpha| < 1$  then the result holds. So suppose  $|\alpha| \geq 1$  and thus  $\alpha \neq 0$ . Now  $f(\alpha) = 0$  gives

$$\alpha = -1 \left( a_1 + \frac{a_2}{\alpha} + \frac{a_3}{\alpha^2} + \cdots + \frac{a_n}{\alpha^{n-1}} \right)$$

and since  $|\alpha| \geq 1$  this gives

$$|\alpha| \leq |a_1| + \frac{|a_2|}{|\alpha|} + \cdots + \frac{|a_n|}{|\alpha|^{n-1}} \leq \sum_{i=1}^n |a_i|.$$

□

Let  $K$  be an arbitrary field,  $f \in K(t)$  rational functions. For  $a \in K$ , we can uniquely write

$$f = (t - a)^n \frac{g(t)}{h(t)}$$

with  $n \in \mathbb{Z}$ ,  $g, h \in K[t]$ , and  $g(a)h(a) \neq 0$ . We define  $\text{ord}_a(f) = n$  to be the order of vanishing of  $f$  at the point  $a$ .

If  $\text{ord}_a(f) > 0$  we say that  $a$  is a **root** of  $f$  with multiplicity  $\text{ord}_a(f)$  and if  $\text{ord}_a(f) < 0$  then  $a$  is a **pole** of order  $\text{ord}_a(f)$ .

Now let  $K = R$  be real closed, then for all  $f \in R(t)$  we can write

$$f(t) = \pm g(t) \prod_{i=1}^r (t - a_i)^{e_i} \tag{1}$$

where  $a_i \in R$ ,  $g(t) \in R(t)$  has no poles in  $R$  and  $g(s) > 0$  for all  $s \in R$ .

**Theorem.** (*Intermediate Value Theorem*) *For  $f \in R(t)$ ,  $a < b$  in  $R$  such that  $f$  has no poles on  $[a, b]$ , and  $f(a)f(b) < 0$ , then the number of roots of  $f$  (counted with multiplicity) in  $[a, b]$  is odd.*

*Proof.* Starting from (1), we may remove any factors from  $f$  corresponding to poles or roots outside of the interval  $[a, b]$  without effecting the Theorem's hypothesis or conclusion. Thus, without loss of generality we can assume

$$f(t) = \prod_{i=1}^r (t - a_i)^{e_i}$$

where  $a_i \in (a, b)$  and  $e_i \in \mathbb{Z}_{>0}$ . Now note that  $a - a_i < 0$  and  $b - a_i > 0$  for all  $i \in [r]$ , which gives  $(a - a_i)(b - a_i) < 0$ . Now computing

$$\begin{aligned} f(a)f(b) &= \prod_{i=1}^r ((a - a_i)(b - a_i))^{e_i} \\ &= \left( \prod_{i=1}^r |(a - a_i)(b - a_i)|^{e_i} \right) (-1)^{\sum_{i=1}^r e_i} \end{aligned}$$

Then since the  $f(a)f(b) < 0$  we must have  $\sum e_i$  is odd, which shows the claim.  $\square$