

Real Closed Fields

Proposition. *Any ordering of a field K extends to the rational function field $K(x_1, \dots, x_n)$.*

Proof. Note that it suffices to prove the claim for $n = 1$. Let $P \subset K$ be a positive cone. Recall by Lemma 1.2.2 from the previous lecture that P does not extend to a positive cone if and only if $-1 \in T_L(P)$. So suppose $-1 \in T_L(P)$, then by definition there exist $a_1, \dots, a_r \in P \setminus \{0\}$ and $f_0, f_1, \dots, f_r \in K[x] \setminus \{0\}$ such that

$$1 + \sum_{i=1}^r a_i \left(\frac{f_i}{f_0} \right)^2 = 0.$$

Clearing the denominators gives

$$f_0^2 + \sum_{i=1}^r a_i f_i^2 = 0$$

however the left hand side has positive degree which is a contradiction. $\square$

Definition. *A real field K is **real closed** if any proper algebraic extension of K is non-real.*

Examples of real closed field include: $\mathbb{R}$ and $\mathbb{R} \cap \overline{\mathbb{Q}}$, where $\overline{\mathbb{Q}}$ denotes the algebraic closure of $\mathbb{Q}$.

Non examples include: $\mathbb{Q}$, $\mathbb{Q}(\sqrt{2})$, $\mathbb{R}(x_1, \dots, x_n)$.

Proposition (1.2.7). *For any field K the following are equivalent*

- (i) *K is real closed.*
- (ii) *K has an ordering that does not extend to any proper algebraic extension.*
- (iii) *K is real, has no proper algebraic extension of odd degree, and every nonzero element of K is equal to $\pm 1 \cdot a^2$ for some $a \in K$.*

Proof. Note that (i) $\implies$ (ii) is clear. For (ii) $\implies$ (iii), let $P \subset K$ be a positive cone that does not extend to any proper algebraic extension. By proposition 1.2.4, K has no proper extension of odd degree, since otherwise P would extend to it. Now observe for any $a \in P$, a has a square root in K since otherwise by proposition 1.2.3, $K(\sqrt{a})$ is a proper algebraic extension of K for which P can be extended to. Now for

$a \in K \setminus P$ we have $-a \in P$ which gives $a = -1 \cdot (\sqrt{-a})^2$. Since $K = P \cup -P$ this completes this implication.

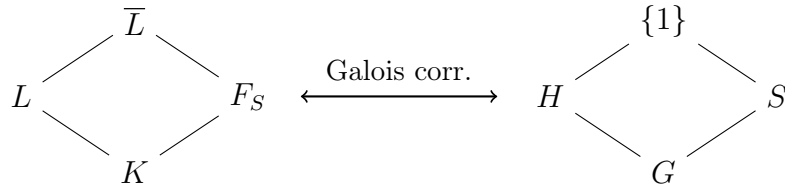
For (iii) $\implies$ (i), note by assumption any non square in K has the form $-a^2$ for some $a \neq 0$. From this it follows that the only quadratic field extension is $K(\sqrt{-1})$. Now any positive cone P on K cannot extend to $K(\sqrt{-1})$ since

$$-1 \in \left(K(\sqrt{-1})\right)^2 \subseteq T_{K(\sqrt{-1})}(P)$$

and Lemma 1.2.2. Thus to prove K is real closed, it suffices to show that any algebraic extension L/K must contain a quadratic extension, which by the above argument shows that L is non real, i.e. it can't be ordered. Thus suppose L/K is a proper algebraic extension. Let $\bar{L}$ be a Galois closure (normal closure) of L/K with Galois group G . Fix a Sylow 2-subgroup $S \subset G$. Note that S exists since $[L : K]$ is even, and divides $[\bar{L} : K]$. By Galois correspondence, there exists a field F_S such that $K < F_S$ and

$$[F_S : K] = [G : S].$$

Now $[G : S]$ is odd since $|S|$ is largest power of 2 that divides $|G|$. Since we assumed K has no proper odd extensions, this forces $[F_S : K] = 1$ which gives $G = S$, i.e. G is a 2-group. Moreover $G_L < G$ is a 2-group, thus by group theory there exists a subgroup $H \supset G_L$, such that $[G : H] = 2$. By Galois correspondence there exists a field F_H , with $K < F_H < L$, such that $[F_H : K] = 2$, i.e. an algebraic extension. This completes the proof. $\square$



Corollary (1.2.8). *If K is real-closed then*

$$P = \{a^2 \mid a \in K\}$$

is the unique positive cone of K .