

Extensions of Orderings

Recall from last time that a subset $T \subseteq K$ of a field K is called a **preorder** if

- $T + T \subseteq T$
- $TT \subseteq T$
- $a^2 \in T$ for all $a \in K$

and T is called **proper** if also $-1 \notin T$.

Corollary (1.1.29). *For any field K , the following are equivalent*

- i) K is real, i.e. K has a positive cone.*
- ii) K has a proper preorder*
- iii) $-1 \notin \Sigma K^2$*
- iv) $a_1^2 + \cdots + a_k^2 = 0$ implies $a_1 = \cdots = a_k = 0$ for all $a_1, \dots, a_k \in K$.*

Proof. $i) \implies ii)$ since any positive cone is a proper preorder. $ii) \implies iii)$ since ΣK^2 is contained in any proper preorder. $iii) \implies iv)$ since if

$$a_1^2 + \cdots + a_k^2 = 0$$

and (w.l.o.g.) $a_1 \neq 0$, we have

$$-1 = \left(\frac{a_2}{a_1}\right)^2 + \cdots + \left(\frac{a_k}{a_1}\right)^2 \in \Sigma K^2$$

which is a contradiction. For $iv) \implies i)$ note that we must have $-1 \notin \Sigma K^2$, since otherwise for some $a_1, \dots, a_k \in K$,

$$1^2 + a_1^2 + \cdots + a_k^2 = 0$$

which contradicts what $iv)$ assumes. Finally recall by the previous lecture that ΣK^2 is then a proper preorder, and that any proper preorder is contained in a positive cone by Zorn's Lemma. Hence K has a positive cone. $\square$

Section 1.2 Real-Closed Fields

Definition. Let (K, P) be an ordered field, and let $L > K$ a field extension. Define

$$T_L(P) := \left\{ \sum_{i=1}^n a_i y_i^2 \mid n \in \mathbb{N}, a_i \in P, y_i \in L \right\}$$

to be the **preorder in L generated by P** .

Note that one can check $T_L(P)$ satisfies the conditions for a preorder of L .

Definition. Let $K < L$ be fields and let $P \subset K$ be a positive cone. We say P **extends** to a positive cone of L if there exists a positive cone $Q \subset L$ such that $Q \cap K = P$.

Lemma (1.2.2). A positive cone $P \subseteq K$ extends to a positive cone of L if and only if $-1 \notin T_L(P)$.

Proof. Observe by the lemma from the previous class since $T_L(P) \subseteq L$ is a preorder, it extends to some positive cone $Q \subseteq L$ if and only if $T_L(P)$ is proper i.e. $-1 \notin T_L(P)$. Thus it suffices to show P extends to a positive cone if and only if $T_L(P)$ is contained in a positive cone. For the first implication note that if Q is a positive cone of L such that $P = Q \cap K$, then for all $y_i \in L$ and $a_i \in P$ we have

$$\sum a_i y_i^2 \in Q$$

thus $T_L(P) \subseteq Q$. For other direction suppose $Q \subseteq L$ is a preorder containing $T_L(P)$. Then since $P \subset T_L(P)$ we have $P \subseteq Q \cap K$. But since $Q \cap K$ is also a preorder we must have $P = Q \cap K$. $\square$

Remark. If $T_L(P)$ is proper then it can be characterized by

$$T_L(P) = \bigcap_{\substack{Q \subseteq L \\ Q \text{ a positive cone} \\ Q \cap K = P}} Q$$

Proof. ($\subseteq$): Note that since $T_L(P)$ is proper, by the proof of the previous lemma it is contained in every positive cone that extends P .

($\supseteq$): Let $a \in L \setminus T_L(P)$. Put $T_1 = T_L(P) - aT_L(P)$, then it follows by the Lemma from the previous lecture that T_1 is a proper preorder and $a \notin T_1$. Consider the family $\{T_\alpha\}$ of all proper preorders that contain $T_L(P)$, and $a \notin T_\alpha$ for all α . By Zorn's lemma there exists a maximal element Q , which by a similar argument to the previous lecture can be shown to be a positive cone such that $a \notin Q$. Thus a is in the complement of the intersection which shows the desired inclusion. $\square$

Proposition. [1.2.3] Let $L = K(\sqrt{a})$ for $a \in K$. Then a positive cone $P \subset K$ extends to L if and only if $a \in P$.

Proof. For the forward direction assume $Q \subseteq L$ extends $P \subseteq K$. Then

$$a = (\sqrt{a})^2 \in Q \cap K = P.$$

For the converse suppose $a \in P$ and assume for contradiction that P does not extend to a positive cone in L . By the previous lemma this implies $-1 \in T_L(P)$. Thus there exist $a_1, \dots, a_n \in P$ and $x_1, \dots, x_n, y_1, \dots, y_n \in K$ such that

$$\begin{aligned} -1 &= \sum_{i=1}^n a_i (x_i + y_i \sqrt{a})^2 \\ &= \sum_{i=1}^n a_i (x_i^2 + ay_i^2 + 2x_i y_i \sqrt{a}) \\ &= \underbrace{\left(\sum_{i=1}^n a_i (x_i^2 + ay_i^2) \right)}_{\in P} + \underbrace{\left(\sum_{i=1}^n 2x_i y_i \right)}_{=0} \sqrt{a}. \end{aligned}$$

Where in the last equality the first term is in P since $a, a_i, x_i^2, y_i^2 \in P$ and the coefficient in front of $\sqrt{a}$ is zero by comparing to -1 . Thus $-1 \in P$, but this contradicts the fact that P is positive cone. Hence P must extend to a positive cone of L . $\square$

Proposition (1.2.4). Let $L > K$ be an extension of K of odd degree, i.e. $[L : K]$ is odd, then every ordering of K extends to L .

Proof. Assume for contradiction that the result does not hold. Let n be the minimal (odd) dimension such that there is an extension $L > K$ with $[L : K] = n$ and a positive cone $P \subseteq K$ that does not extend to L . By the Primitive Element Theorem, there exists an irreducible $f(x) \in K[x]$ such that $\deg f = n$ and

$$L \cong K[x]/\langle f \rangle = K(\alpha)$$

where $\alpha = x \bmod \langle f \rangle$. By Lemma 1.2.2., if $T_L(P)$ is a proper preorder then P extends to L , so we must have $-1 \in T_L(P)$. Thus there exist $a_1, \dots, a_r \in P$ and $y_1, \dots, y_r \in L$ such that

$$1 + \sum_{i=1}^r a_i y_i^2 = 0. \quad (\star)$$

Without loss of generality assume $a_i \neq 0$ and $y_i \neq 0$ for all $i \in [r]$. Now for all i let $g_i(x) \in K[x]$ be such that

$$y_i = g_i(\alpha) \quad \text{and} \quad \deg(g_i) < n.$$

Now we can lift $(\star)$ up to $K[x]$ so that

$$1 + \sum_{i=1}^r a_i (g_i(x))^2 = f(x)h(x) \quad (\star\star)$$

for some $h(x) \in K[x]$. Note that

$$\deg \left(1 + \sum_{i=1}^r a_i (g_i(x))^2 \right) = 2d$$

where $d = \max_i \{\deg(g_i)\}$. Hence $(\star\star)$ gives

$$2d = \deg(f) \deg(h)$$

and since $\deg(f) = n$ is odd, $\deg(h)$ must be odd, moreover

$$\deg(h) = 2d - n < d < n.$$

Now let $h_1(x) \in K[x]$ be an irreducible factor of $h(x)$ of odd degree $< n$. Define

$$L_1 := K[x]/\langle h_1 \rangle.$$

Then L_1 is an extension of K of odd degree strictly less than n . Furthermore since $f(x)h(x) = 0 \pmod{\langle h_1 \rangle}$, equation $(\star\star)$ gives $-1 \in T_{L_1}(P)$ i.e. P does not extend to a positive cone in L_1 . Hence $L_1 > K$ is an odd extension of degree strictly less than n where the result does not hold. This contradicts the minimality of n . Thus the result holds for all odd extensions. $\square$