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Lecture 5: Putinar's Positivstellensatz (Theobald §6.7)

1 Setup

Definition 1. *A subset $M \subseteq \mathbb{R}[\mathbf{x}]$ is called a* quadratic module *if:*

1. $1 \in M$;
2. $M + M \subseteq M$;
3. $a^2 M \subseteq M$ *for every* $a \in \mathbb{R}[\mathbf{x}]$.

The set of all sums of squares in $\mathbb{R}[\mathbf{x}]$ is denoted by $\sum \mathbb{R}[\mathbf{x}]^2 = \{a_1^2 + \cdots + a_m^2 \mid a_i \in \mathbb{R}[\mathbf{x}]\}$

Given $g_1, \ldots, g_m \in \mathbb{R}[\mathbf{x}]$, define

$$\mathrm{QM}(g_1, \ldots, g_m) = \left\{ \sigma_0 + \sigma_1 g_1 + \cdots + \sigma_m g_m \;\middle|\; \sigma_i \in \sum \mathbb{R}[\mathbf{x}]^2 \right\}.$$

It is the smallest quadratic module containing $g_1, \ldots, g_m$.

Let $g_1, \ldots, g_m \in \mathbb{R}[\mathbf{x}] = \mathbb{R}[x_1, \ldots, x_n]$. We say that $g_1, \ldots, g_m$ satisfy *Putinar's property* if every polynomial that is strictly positive on $S(g_1, \ldots, g_m)$ belongs to $\mathrm{QM}(g_1, \ldots, g_m)$.

A quadratic module M is said to be *Archimedean* if

$$\exists\, N \in \mathbb{N} \quad \text{such that} \quad N - \sum_{i=1}^{n} x_i^2 \in M.$$

Remark 1. *If $\mathrm{QM}(g_1, \ldots, g_m)$ is Archimedean, then $S(g_1, \ldots, g_m)$ is compact. On the other hand, if $S(g_1, \ldots, g_m)$ is compact, then there exists $N \in \mathbb{N}$ such that*

$$S(g_1, \ldots, g_m) \subseteq S\left(N - \sum_{i=1}^{n} x_i^2\right).$$

Consequently,

$$S(g_1, \ldots, g_m) = S\left(g_1, \ldots, g_m, N - \sum_{i=1}^{n} x_i^2\right).$$

Example 1. *Consider*

$$g = (1 - x^2)^3, \qquad f = 1 - x^2 \in \mathbb{R}[x].$$

Claim: $\mathrm{QM}(g)$ *is Archimedean. Indeed,*

$$2 - x^2 = \frac{2}{3} + \frac{4}{3}\left(x^3 - \frac{3}{2}x\right)^2 + \frac{4}{3}(1 - x^2)^3.$$

Thus,

$$2 - x^2 = \sigma_0 + \sigma_1 g,$$

where

$$\sigma_0 = \frac{2}{3} + \frac{4}{3}\left(x^3 - \frac{3}{2}x\right)^2, \qquad \sigma_1 = \frac{4}{3}.$$

Hence $2 - x^2 \in \mathrm{QM}(g)$*, so* $\mathrm{QM}(g)$ *is Archimedean. Moreover,*

$$S(g) = [-1, 1].$$

The polynomial

$$f = 1 - x^2$$

is nonnegative on $S(g)$*, but* $f \notin \mathrm{QM}(g)$*. Suppose, for contradiction, that* $f \in \mathrm{QM}(g)$*. Then*

$$1 - x^2 = \sigma_0 + \sigma_1 g,$$

where σ_0 *and* σ_1 *are sums of squares. Since* σ_0 *is a sum of squares, write*

$$\sigma_0 = a_1^2 + \cdots + a_k^2.$$

Evaluating at $x = 1$ *gives*

$$\sigma_0(1) = 0.$$

Therefore,

$$a_i(1) = 0$$

for every i*. Hence each* $a_i(x)$ *is divisible by* $x - 1$*, so*

$$a_i(x)^2 = b_i(x)^2 (x - 1)^2.$$

Consequently,

$$\sigma_0 = (x - 1)^2 \left(b_1^2 + \cdots + b_k^2\right),$$

and therefore σ_0 *vanishes to order at least* 2 *at* $x = 1$*. Also, since*

$$g = (1 - x^2)^3,$$

the term $\sigma_1 g$ vanishes to order at least 3 at $x = 1$. Thus, the right-hand side

$$\sigma_0 + \sigma_1 g$$

vanishes to order at least 2 at $x = 1$. However,

$$1 - x^2 = (1 - x)(1 + x)$$

vanishes to order exactly 1 at $x = 1$, which is a contradiction. Therefore,

$$f = 1 - x^2 \notin \mathrm{QM}(g).$$

2 Pulinar's Theorem

Theorem 1. *If $g_1, \ldots, g_m \in \mathbb{R}[\mathbf{x}]$ are such that $\mathrm{QM}(g_1, \ldots, g_m)$ is Archimedean then $g_1, \ldots, g_m$ have Putinar's property.*

Lemma 1 (Lemma 6.43)**.** *If $S(g_1, \ldots, g_m)$ is compact and $g_1, \ldots, g_m$ have Putinar's property, then, for any $g_{m+1} \in \mathbb{R}[x]$, the polynomials*

$$g_1, \ldots, g_m, g_{m+1}$$

have Putinar's property.

Example 2 (Example 6.45)**.** *For any $N \in \mathbb{Z}_{>0}$, the polynomial*

$$N - \sum_{i=1}^{n} x_i^2 \in \mathbb{R}[x_1, \ldots, x_n]$$

has Putinar's property.

Proof. Assume that $\mathrm{QM}(g_1, \ldots, g_m)$ is Archimedean. Then there exists $N \in \mathbb{N}$ such that

$$N - \sum_{i=1}^{n} x_i^2 \in \mathrm{QM}(g_1, \ldots, g_m).$$

By (6.4.5), the polynomial

$$N - \sum_{i=1}^{n} x_i^2$$

has Putinar's property. Hence, by (6.4.3),

$$g_1, \ldots, g_m,\ N - \sum_{i=1}^{n} x_i^2$$

have Putinar's property. Moreover, since

$$N - \sum_{i=1}^{n} x_i^2 \in \mathrm{QM}(g_1, \ldots, g_m),$$

adding this polynomial as an additional generator does not change the quadratic module. Therefore,

$$\mathrm{QM}\left(g_1, \ldots, g_m, N - \sum_{i=1}^{n} x_i^2\right) = \mathrm{QM}(g_1, \ldots, g_m).$$

Also,

$$S(g_1, \ldots, g_m) = S\left(g_1, \ldots, g_m, N - \sum_{i=1}^{n} x_i^2\right).$$

Thus, if $f > 0$ on $S(g_1, \ldots, g_m)$, then

$$f \in \mathrm{QM}\left(g_1, \ldots, g_m, N - \sum_{i=1}^{n} x_i^2\right) = \mathrm{QM}(g_1, \ldots, g_m).$$

Hence $g_1, \ldots, g_m$ have Putinar's property. $\square$