

Preorders

Recall from last time: an ordered field $(K, \leq)$ is **Archimedean** if for all $a, b \in K$ with $b > 0$, there exists an $n \in \mathbb{N}$ such that $n \cdot b > a$.

Also recall **Holder's Theorem**: If (K, P) is an Archimedean ordered field, there exists a field embedding $\phi : K \rightarrow \mathbb{R}$ where $\phi(P) \subset \mathbb{R}_{\geq 0}$ and this map is uniquely determined.

Recap of proof. For $a \in K$, define $I_a = (-\infty, a] \cap \mathbb{Q}$ and $J_a = [a, \infty) \cap \mathbb{Q}$. It turns out that we require

$$\phi(a) = \sup I_a = \inf J_a.$$

We've checked that ϕ is additive and multiplicative. We can also observe that

$$\phi(1) = \sup I_1 = \inf J_1 = 1.$$

Finally, observe that if $a \leq_P b$, then we have $J_b \subset J_a$ and therefore $\phi(a) = \inf J_a \leq \inf J_b = \phi(b)$. Apply this to $a = 0$ to obtain that for all $b \in P$, we have $\phi(b) \in \mathbb{R}_{\geq 0}$, and we are done. $\square$

Definition (1.1.23). A **preorder** of a field K is a subset $T \subset K$ such that

$$(i) \quad T + T \subset T, TT \subset T$$

$$(ii) \quad a^2 \in T, \quad \forall a \in K.$$

T is said to be **proper** if $-1 \notin T$.

Comments:

- Positive cones are proper preorders.
- If T is improper, $-1 \in T$ and $1 \in T$. For any $x \in K$, we then have

$$x = \left(\frac{x+1}{2} \right)^2 - \left(\frac{x-1}{2} \right)^2 \in T$$

by preorder properties (i) and (ii), therefore $T = K$.

- If $\{T_i\}_{i \in I}$ is a family of preorders of K , then $\bigcap_{i \in I} T_i$ is also a preorder.
- $T = \sum K^2$ is the smallest preorder in any field K .

- If $\{T_i\}_{i \in I}$ is a family of preorders of K which is upward-closed (i.e. for all $i_1, i_2 \in I$ there exists an $i_3 \in I$ such that $T_{i_1} \cup T_{i_2} \subset T_{i_3}$) then $\bigcup_{i \in I} T_i$ is a preorder.
- For $S \subset K$,

$$PO(S) = \left\{ \sum_{\vec{e} \in \{0,1\}^r} S_{\vec{e}} \cdot t_1^{e_1} t_2^{e_2} \dots t_r^{e_r} : r \in \mathbb{N}, t_1, \dots, t_r \in S \cup \{1\}, S_{\vec{e}} \in \sum K^2 \right\}$$

is the **preorder generated by S** and one can check that this is the smallest preorder of K containing S .

Lemma. *If $T \subset K$ is a proper preorder and $a \in K \setminus T$, then $T - aT = \{s - at : s, t \in T\}$ is again a proper preorder of K , which contains T .*

Note: if $a = -1$, then $T - aT = T$.

Proof. (It is easy to check that $T - aT$ is a preorder of K). Now, suppose that $-1 \in T - aT$. Then, there exist $s, t \in T$ such that $-1 = s - at$. Since T is proper this requires $t \neq 0$, hence $a = (s + 1)/t = (1/t^2)(t(s + 1)) \in T$, contradicting that $a \in K \setminus T$. Thus $-1 \notin T - aT$ so it is a proper preorder. $\square$

Proposition. *Any proper preorder is contained in some positive cone.*

Proof. Fix a preorder $T \subset K$. Let $\mathcal{T}$ be the family of all preorders containing T . Note that $\mathcal{T}$ is a poset whose chains are upward closed, hence every chain has an upper bound in $\mathcal{T}$. By Zorn's Lemma, we then have that $\mathcal{T}$ has a maximal element P . To show that P is a positive cone, it is enough to show that $P \cup (-P) = K$. Let $a \in K$. If $a \in P$, we are done. If $a \notin P$, then $P - aP$ is a preorder containing P , but P is maximal so $P = P - aP$. Therefore $-a \in P - aP = P$ so $a \in -P$, and we are done. $\square$