

Ordered Fields (cont.)

Let (K, P) be an ordered field and $P \subset K$ be a **positive cone**, i.e.

1. $P + P \subset P, PP \subset P$
2. $P \cup (-P) = K$
3. $P \cap (-P) = \{0\}$

and $a \leq_P b \iff b - a \in P$ be the corresponding total order.

For $a \in K$, define

$$\operatorname{sgn}_P(a) = \begin{cases} 1 & \text{if } a \in P \setminus \{0\} \\ -1 & \text{if } a \in -P \setminus \{0\} \\ 0 & \text{if } a = 0 \end{cases}$$

Remark 1. We can have different orderings on the same field.

Example 1. Consider $K = \mathbb{Q}(\sqrt{2})$, and the two embeddings $\mathbb{Q}(\sqrt{2}) \xrightarrow{\phi_1} \mathbb{R}$ and $\mathbb{Q}(\sqrt{2}) \xrightarrow{\phi_2} \mathbb{R}$ by

$$\phi_1(a + b\sqrt{2}) = a + b\sqrt{2}$$

and

$$\phi_2(a + b\sqrt{2}) = a - b\sqrt{2}.$$

Then, one can see that $P = \phi_1^{-1}(\mathbb{R}_{\geq 0})$ and $Q = \phi_2^{-1}(\mathbb{R}_{\geq 0})$ are distinct positive cones in K .

Note some definitions we can make within the context of an ordered field:

- **Intervals:** $[a, b]_P \subset K$, and other forms of intervals, have the same meaning as in the field $\mathbb{R}$
- **Absolute value:** Define $|a|_P = \operatorname{sgn}_P(a)a$
- **Triangle inequality:** Fix any $a, b \in K$. If $ab \geq 0$, then $|a + b|_P = |a|_P + |b|_P$. If $ab \leq 0$, then $|a + b|_P \leq \max(|a|_P, |b|_P) \leq |a|_P + |b|_P$.

If (K, P) is an ordered field and $K_0 \subset K$ is a subfield, then $(K_0, K_0 \cap P)$ is an ordered field. We say (K, P) is an **order extension** of $(K_0, K_0 \cap P)$.

Exercise 1. If P and Q are positive cones in K such that $P \subset Q$, then $P = Q$.

A consequence of the above exercise is that if for an ordered field K we have $\sum K^2$ is a positive cone (which is true for $\mathbb{R}$, for example) then it is the only positive cone in K .

Definition 1 (Definition 1.1.16). An ordered field $(K, \leq)$ is **Archimedean** if for all $a, b \in K$, if $b > 0$ then there exists an $n \in \mathbb{N}$ such that $n \cdot b > a$.

For example, the field $R(x)$ with positive cone

$$P = \left\{ x^n \frac{p(x)}{q(x)} : p(0)q(0) > 0 \right\} \cup \{0\}$$

is not Archimedean: notice that for any real number $\varepsilon > 0$, we have $\frac{\varepsilon - x}{1} \in P$ by definition of P , and therefore $\varepsilon > x$.

Theorem 1 (Holder's Theorem). If (K, P) is an Archimedean ordered field, then there exists a unique field homomorphism $\phi : K \rightarrow \mathbb{R}$ such that $\phi(P) \subset \mathbb{R}_{\geq 0}$. Such a map is called an **order embedding**.

Corollaries:

- The only embedding from $\mathbb{R}$ to itself is the identity.
- If $K \supsetneq \mathbb{R}$ is a proper extension, then K does not have an Archimedean embedding.
 - Indeed: $\mathbb{R} \subsetneq K$ so if $\phi : K \rightarrow \mathbb{R}$ is an Archimedean embedding, then $\phi|_{\mathbb{R}} = \text{id}$ so ϕ cannot be injective, contradicting Holder's Theorem.

Proof (of Holder's Theorem). For any $a \in K$, define

$$I_a = (-\infty, a] \cap \mathbb{Q}$$

and

$$J_a = [a, \infty) \cap \mathbb{Q}.$$

Note that since $\mathbb{Q} \xrightarrow{\text{id}} \mathbb{R}$ is an order embedding, we can think of I_a and J_a as subsets of $\mathbb{R}$. Now, for all $x \in I_a$ and $y \in J_a$ we have $x \leq y$, and also $\mathbb{Q} = I_a \cup J_a$. Thus, if $\phi : K \rightarrow \mathbb{R}$ is an order embedding, we must have that

$$\phi(a) = \sup I_a = \inf J_a$$

i.e. the definition of ϕ is forced. Note that since K is Archimedean, I_a and J_a are nonempty. Now, notice that for $a, b \in K$, $I_a + I_b \subset I_{a+b}$ and therefore $\phi(a) + \phi(b) \leq \phi(a+b)$. However also $J_a + J_b \subset J_{a+b}$, so $\phi(a) + \phi(b) \geq \phi(a+b)$, and thus $\phi(a) + \phi(b) = \phi(a+b)$. Define $I_a^+ = I_a \cap \mathbb{R}_{\geq 0}$ and $J_a^+ = J_a \cap \mathbb{R}_{\geq 0}$. Then,

$$I_a^+ I_b^+ \subset I_{ab}^+$$

and

$$J_a^+ J_b^+ \subset J_{ab}^+$$

Therefore ϕ is multiplicative on P , hence multiplicative. It remains to show that $\phi(1_k) = 1_{\mathbb{R}}$ and $\phi(P) \subset \mathbb{R}_{\geq 0}$. $\square$