

## Lecture 7: Orderings of Fields

**Definition.** An *ordered field* is a pair  $(K, \leq)$  where  $K$  is a field and  $\leq$  is a total order on  $K$  such that

$$1. a \leq b \Rightarrow a + c \leq b + c \quad \forall a, b, c \in K$$

$$2. a \leq b \Rightarrow ac \leq bc \quad \forall a, b, c \in K, c \geq 0$$

**Proposition.** Let  $(K, \leq)$  be an ordered field. The set  $P = \{a \in K : a \geq 0\}$  satisfies

$$(a) P + P \subset P, PP \subset P$$

$$(b) P \cup (-P) = K$$

$$(c) P \cap (-P) = \{0\}$$

$$(c') -1 \notin P$$

Conversely, if  $P \subset K$  satisfies (a), (b), and (c) or (c'), then this gives an ordering  $\leq_P$  on  $K$  (i.e.  $(K, \leq_P)$  is an ordered field with nonnegative elements  $P$ ).

**Remark.** A set  $P$  satisfying such conditions is called a **positive cone**.

**Lemma 1.** Let  $(K, \leq)$  be an ordered field. Fix some element  $a \in K$ . Then,

- $a \geq 0 \iff -a \leq 0$
- $a \geq 0 \iff \frac{1}{a} \geq 0$ , if  $a \neq 0$
- $a^2 \geq 0$

Assuming this lemma we will prove the above proposition.

*Proof.* We first show that total orderings give rise to positive cones, i.e. that  $P = \{a \in K : a \geq 0\}$  satisfies the definition of a positive cone. First note that (1) and (2) imply (a) directly. For (b), observe that for any  $a \in K \setminus P$  then  $a < 0$  so  $a \leq 0$  so by the lemma  $-a \geq 0$ , hence  $-a \in P$ . For (c), observe that if  $a \geq 0$  and  $-a \geq 0$ , then the lemma gives that  $a \leq 0$ , so by the totality of the order  $\leq$  we have  $a = 0$ . Finally, (c') holds for  $P$  since  $1 = 1^2 \geq 0$  (by the lemma) therefore  $1 \in P$  so  $-1 \notin P$  by (c).

We now will show that positive cones give rise to orderings, i.e. they allow us to define a total ordering on  $K$ . We assume that a subset  $P \subset K$  satisfies (a), (b), and (c). Define a relation  $\leq_P$  on  $K$  by  $a \leq_P b$  if and only if  $b - a \in P$ . We will verify that  $(K, \leq_P)$  satisfies (1) and (2) and that the order  $\leq_P$  is total. For (1), observe that

$$\begin{aligned} a \leq_P b &\iff b - a \in P \\ &\iff (b + c) - (a + c) \in P \\ &\iff b + c \leq_P a + c \end{aligned}$$

For (2), observe that for  $a, b, c \in K$  with  $a \leq_P b$  and  $0 \leq_P c$ , by (a) since  $c \in P$  we have  $cb - ca = c(b - a) \in P$ , i.e.  $ca \leq_P cb$ . To show that  $\leq_P$  is a total order, note that for any  $a, b \in K$  either  $a - b \in P$  or  $b - a \in P$  by property (b), hence  $a \leq_P b$  or  $b \leq_P a$ . Furthermore property (c) implies that  $a \leq_P b$  and  $b \leq_P a$  if and only if  $b - a = 0$  i.e.  $b = a$ , establishing that  $\leq_P$  is total. We have shown that (a), (b), and (c) imply (1) and (2), so it remains to show that (a), (b) and (c') imply (1) and (2). We do so by claiming (a), (b), (c) are equivalent to (a), (b), (c') :

- Assume (a), (b), (c). Then, because  $1 = 1^2 \in P$ , (c) implies that  $-1 \notin P$ , hence (c') holds.
- Assume (a), (b), (c'), but also that (c) doesn't hold so there is a nonzero element  $a \in P \cap -P$ . Then, property (a) gives us that both

$$-a^2 = a \cdot (-a) \in P$$

and

$$\frac{1}{a^2} = \left(\pm \frac{1}{a}\right)^2 \in P$$

and therefore  $-1 = (-a^2)(\pm 1/a)^2 \in P$ , contradicting (c').

Thus (a), (b), (c) and (a), (b), (c') are equivalent and we are done.  $\square$

**Example 1.**  $\mathbb{R}$  (with usual ordering) as well as any of its subfields (with restriction of usual ordering)

**Example 2.** If  $K$  is algebraically closed, then  $x^2 + 1 \in K[x]$  has a root in  $K$ , hence  $-1$  is a square in  $K$ . Therefore it belongs to all positive cones of  $K$ , meaning by our above proposition,  $K$  has no orderings.

**Corollary** (Corollary 1.1.8). For any field  $K$  with a positive cone  $P \subset K$ , we have  $\sum K^2 \subset P$ , where

$$\sum K^2 = \{a_1^2 + \cdots + a_k^2 : a_1, \dots, a_k \in K\}$$

is the set of sums-of-squares in  $K$ .

**Note:** If  $K$  is a field which can be ordered, we say  $K$  is a **real field**. Thus, by Corollary 1.1.8, if  $K$  is real then  $-1 \notin \sum K^2$ , which moreover implies that  $\text{char}(K) = 0$ .

**Example 3.** Let  $K = \mathbb{R}(x)$  be the field of fractions of  $\mathbb{R}[x]$ . We write

$$K = \left\{ x^n \frac{p(x)}{q(x)} : n \in \mathbb{Z}, p, q \in \mathbb{R}[x], q(0) \neq 0, p(0) \neq 0 \right\} \cup \{0\}$$

and claim that

$$P = \left\{ x^n \frac{p(x)}{q(x)} \in K : p(0)q(0) > 0 \right\} \cup \{0\}.$$

is a positive cone in  $K$ .

*Proof.* Consider  $x^n \frac{p_1(x)}{q_1(x)}, x^m \frac{p_2(x)}{q_2(x)} \in P$  nonzero, and without loss of generality assume  $n \geq m$ . Then,

$$x^n \frac{p_1}{q_1} + x^m \frac{p_2}{q_2} = x^m \left( x^{n-m} \frac{p_1}{q_1} + \frac{p_2}{q_2} \right) = x^m \left( x^{n-m} \frac{p_1 q_2 + p_2 q_1}{q_1 q_2} \right).$$

Note that if  $n > m$ ,  $\text{sgn} \left( \frac{\text{num}(0)}{\text{denom}(0)} \right) = \text{sgn} \left( \frac{q_1(0)p_2(0)}{q_1(0)q_2(0)} \right) = 1$  and if  $n = m$ ,  $\text{sgn} \left( \frac{q_1(0)p_2(0)}{q_1(0)q_2(0)} \right) = 1$  so  $\text{sgn} \left( \frac{\text{num}(0)}{\text{denom}(0)} \right) = 1$ , thus  $P + P \subset P$ . The remaining conditions to fulfill the definition of a positive cone can quickly be checked.  $\square$