MATH 8102 Fall 2026 Scribe: Elena Bou Aoun
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Lecture 4: Handelman's Theorem (Theobald §6.6)

1 Setup

Let $g_1, \dots, g_m \in \mathbb{R}[\mathbf{x}]$ and $\beta = (\beta_1, \dots, \beta_m) \in (\mathbb{Z}_{\geq 0})^m$. Let g^β denote the element $g_1^{\beta_1} \cdots g_m^{\beta_m} \in \mathbb{R}[\mathbf{x}]$.

The *basic closed semialgebraic set* determined by $g_1, \dots, g_m$ is

$$\mathcal{S}(g_1, \dots, g_m) := \left\{ \mathbf{x} \in \mathbb{R}^n \mid g_i(\mathbf{x}) \geq 0 \ \forall i \right\}.$$

The *semiring* generated by $g_1, \dots, g_m$ is

$$\mathbb{R}_{\geq 0}[g_1, \dots, g_m] := \left\{ \sum_\beta c_\beta g^\beta \mid c_\beta \geq 0 \ \forall \beta \right\}.$$

When $g_1, \dots, g_m$ are affine-linear, $\mathcal{S}(g_1, \dots, g_m)$ is a polyhedron. A compact polyhedron is called a *polytope*.

2 Handelman's Theorem

Theorem 1 (Handelman's Theorem)**.** *Let $g_1, g_2, \dots, g_m \in \mathbb{R}[\mathbf{x}]$ be affine linear polynomials such that $\mathcal{S}(g_1, g_2, \dots, g_m)$ is a nonempty polytope. Any polynomial $p \in \mathbb{R}[\mathbf{x}]$ which is strictly positive on $\mathcal{S}(g_1, g_2, \dots, g_m)$ must belong to $\mathbb{R}_{\geq 0}[g_1, g_2, \dots, g_m]$.*

Example 1. *Let $g_i(\mathbf{x}) = b_i - \mathbf{a}_i^T \mathbf{x}$ and $p(\mathbf{x}) = d - \mathbf{c}^T \mathbf{x}$. Observe*

$$\begin{aligned}
p(\mathbf{x}) > 0 \text{ on } \mathcal{S}(g_1, \dots, g_m) &\iff \mathcal{S}(g_1, \dots, g_m, -p) = \emptyset \\
&\iff \begin{pmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_m & -\mathbf{c} \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_m \\ y_{m+1} \end{pmatrix} = \mathbf{0}, \qquad \textit{(by Farkas' Lemma)} \\
&\qquad \begin{pmatrix} b_1 & \cdots & b_m & -d \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_m \\ y_{m+1} \end{pmatrix} = -1, \qquad \begin{pmatrix} y_1 \\ \vdots \\ y_m \\ y_{m+1} \end{pmatrix} \in \mathbb{R}_{\geq 0}^{m+1}.
\end{aligned}$$

Notice that if $\mathcal{S}(g_1,\ldots,g_m)\neq\emptyset$*, we must have* $y_{m+1}>0$*.*

Hence, $p(\mathbf{x})=\dfrac{1}{y_{m+1}}\left(1+\sum_{i=1}^{m}y_ig_i(\mathbf{x})\right)\in\mathbb{R}_{\geq 0}[g_1,\ldots,g_m]$*.*

Proof of Handelman's Theorem. By translating $\mathcal{S}(g_1,g_2,\ldots,g_m)$, we may assume that $\mathcal{S}(g_1,g_2,\ldots,g_m)$ is contained in the interior of $(\mathbb{R}_{\geq 0})^n$; the statement of the theorem is unaffected. By Farkas' Lemma, it is enough to show that

$$p\in\mathbb{R}_{\geq 0}[g_1,g_2,\ldots,g_m,\ell_1,\ldots,\ell_{n+1}],$$

where the ℓ_i are affine-linear polynomials that are strictly positive on $\mathcal{S}(g_1,g_2,\ldots,g_m)$. Indeed, Farkas' Lemma implies that each ℓ_i belongs to the semiring $\mathbb{R}_{\geq 0}[g_1,\ldots,g_m]$ in this case—see Example 1.

Since we have translated $\mathcal{S}(g_1,g_2,\ldots,g_m)$ into the interior of $(\mathbb{R}_{\geq 0})^n$, we have

$$\ell_i=x_i>0\qquad\text{for all } i=1,\ldots,n$$

on $\mathcal{S}(g_1,g_2,\ldots,g_m)$. Hence, set

$$\ell_i(\mathbf{x})=x_i,\qquad i=1,\ldots,n.$$

and

$$\ell_{n+1}(\mathbf{x}):=\tau-\sum_{j=1}^{m}g_j(\mathbf{x})-\sum_{i=1}^{n}\ell_i(\mathbf{x})=\tau-\sum_{j=1}^{m}g_j(\mathbf{x})-\sum_{i=1}^{n}x_i,$$

where $\tau\in\mathbb{R}_{>0}$ is chosen sufficiently large so that

$$\ell_{n+1}(\mathbf{x})>0\qquad\text{for all } \mathbf{x}\in\mathcal{S}(g_1,g_2,\ldots,g_m).$$

Now, for any positive constant c, define the polynomial

$$h_c(x_1,\ldots,x_{m+n+1})=p(x_1,\ldots,x_n)+c\sum_{j=1}^{m}\big(x_{n+j}-g_j(\mathbf{x})\big)$$

where $x_{n+1},\ldots,x_{m+n+1}$ are $m+1$ new variables. Note that

$$h(x_1,\ldots,x_n,g_1(\mathbf{x}),\ldots,g_m(\mathbf{x}),x_{n+m+1})=p(\mathbf{x}).$$

Let

$$\Delta=\left\{\mathbf{x}\in\mathbb{R}_{\geq 0}^{n+m+1}:\sum_{i=1}^{n+m+1}x_i=\tau\right\}$$

be the scaled standard simplex, and let

$$\Delta' = \left\{\mathbf{x} \in \Delta : x_{n+1} = g_1(\mathbf{x}), \ldots, x_{n+m} = g_m(\mathbf{x})\right\}.$$

For any $\mathbf{x} \in \Delta'$ and $j \in \{1, \ldots, m\}$, we have $g_j(\mathbf{x}) = x_{n+j} \geq 0$. Hence, for each $c \gg 0$, the polynomial h_c is strictly positive on some open set $U \subseteq \Delta$ with $\Delta' \subseteq U \subseteq \Delta$. Now, $\Delta \setminus U$ is compact, so p attains a minimum $p_{\min}$ on $\Delta \setminus U$, and $f(\mathbf{x}) := \sum_{j=1}^{m} \left(x_{n+j} - g_j(\mathbf{x})\right)$ attains a minimum $f_{\min} > 0$ on $\Delta \setminus U$. We can therefore choose $c > \dfrac{-p_{\min}}{f_{\min}}$ so that h_c is strictly positive on Δ. Let

$$\widetilde{h} = \left(\frac{1}{\tau}\sum_i x_i\right)^{\deg p} h\left(\frac{x_1}{\frac{1}{\tau}\sum_i x_i}, \ldots, \frac{x_{n+m+1}}{\frac{1}{\tau}\sum_i x_i}\right)$$

be the homogenization of h with respect to $\frac{1}{\tau}\sum_i x_i$. Since $\widetilde{h}$ is strictly positive on Δ, Pólya's Theorem gives an $N \in \mathbb{N}_0$ such that all coefficients of $H(\mathbf{x}) := \left(\frac{1}{\tau}\sum_i x_i\right)^N \widetilde{h}(\mathbf{x})$ are nonnegative. Substituting successively $x_{n+m+1} = \ell_{n+1}$ and $x_{n+j} = g_j(\mathbf{x})$, $1 \leq j \leq m$, we see that

$$p(x_1, \ldots, x_n) = H(x_1, \ldots, x_n, g_1, \ldots, g_m, \ell_{n+1}),$$

which gives $p \in \mathbb{R}_{\geq 0}[g_1, \ldots, g_m, \ell_1, \ldots, \ell_n, \ell_{n+1}]$, as needed.

$\square$

Example 2. *The hypothesis of compactness in Theorem 1 is necesary. Consider $g_1(x) = x - 1$. Then $\mathcal{S}(g_1) = \{x \in \mathbb{R} : x - 1 \geq 0\} = [1, \infty)$, which is not compact. Let*

$$p(x) = (x-1)^2 - (x-1) + 1 = x^2 - 3x + 3.$$

For $x \in \mathcal{S}(g_1)$, let $t = x - 1 \geq 0$. Then $p(x) = t^2 - t + 1 = \left(t - \frac{1}{2}\right)^2 + \frac{3}{4} > 0$. Hence, p is strictly positive on $\mathcal{S}(g_1)$. However, $p(x) = 1 - (x-1) + (x-1)^2$, which has a negative coefficient in front of $(x-1)$. Since the representation of p as a polynomial in $(x-1)$ is unique, we have $p \notin \mathbb{R}_{\geq 0}[x-1] = \mathbb{R}_{\geq 0}[g_1]$.

Thus, without compactness, $p > 0$ on $\mathcal{S}(g_1)$ does not necessarily imply $p \in \mathbb{R}_{\geq 0}[g_1]$.

Example 3. *Consider $[-1, 1] = \mathcal{S}(g_1, g_2)$, where $g_1(x) = 1 + x$ and $g_2(x) = 1 - x$, and $p(x) = x^2 + 1$. Note that the parabola $x^2 + 1$ is strictly positive on $[-1, 1] = \mathcal{S}(g_1, g_2)$, but it cannot be represented as a nonnegative linear combination of the monomials 1, $1 + x$, $1 - x$, and $(1 + x)(1 - x)$ in the g_i.*