MATH 8102 Fall 2026 — Scribe: Marc Abi Aad
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Lecture 3: Pólya's Theorem (Theobald §6.6)

Setup: For any $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$, let $\mathbf{x}^\alpha$ denote the element $x_1^{\alpha_1} \cdot \dots \cdot x_n^{\alpha_n}$ in $\mathbb{R}[x_1, \dots, x_n] = \mathbb{R}[\mathbf{x}]$. In this case, $\deg(\mathbf{x}^\alpha) = |\alpha| = \sum_{i=1}^n \alpha_i$, and $f = \sum c_\alpha \mathbf{x}^\alpha \in \mathbb{R}[\mathbf{x}]$ is homogeneous if $\deg(\mathbf{x}^\alpha)$ is constant over nonzero c_α.

Theorem 1 (Pólya's Theorem)**.** *Let $P(\mathbf{x}) \in \mathbb{R}[x_1, \dots, x_n]$ be a homogeneous polynomial such that $P(\mathbf{x}) > 0$ for all $\mathbf{x} \in \mathbb{R}^n_{\geq 0} \setminus \{\mathbf{0}\}$ (equivalently, P is strictly positive on the unit simplex). Then for all integers N sufficiently large, the polynomial*

$$(x_1 + \dots + x_n)^N P(\mathbf{x}) \tag{1}$$

has nonnegative coefficients (with respect to the basis of standard monomials $\{x^\alpha\}_{\alpha \in \mathbb{Z}^n_{\geq 0}}$).

Proof. Set $P(\mathbf{x}) = \sum_{|\alpha| = d} c_\alpha \mathbf{x}^\alpha \Rightarrow d = \deg P$.

We expand (1) using the multinomial theorem:

$$\begin{aligned}
(x_1 + \dots + x_n)^N P(\mathbf{x}) &= \sum_{|\alpha|=d} \sum_{|\gamma|=N} \left(c_\alpha \binom{N}{\gamma_1, \dots, \gamma_N} \mathbf{x}^{\alpha+\gamma} \right) \\
&= \sum_{|\beta|=d+N} \sum_{\substack{|\alpha|=d \\ \alpha \leq \beta}} \left(c_\alpha \binom{N}{\beta_1 - \alpha_1, \dots, \beta_n - \alpha_n} x^\beta \right) \\
&\qquad\qquad (\ \beta = \alpha + \gamma, \text{ change summation order}) \\
&= \sum_{|\beta|=d+N} \left[\frac{N!}{\beta_1! \dots \beta_n!} \sum_{|\alpha|=d} c_\alpha \prod_{i=1}^n \beta_i(\beta_i - 1) \cdot \dots \cdot (\beta_i - (\alpha_i - 1)) \right] x^\beta.
\end{aligned}$$

(Note: $\binom{N}{\gamma_1, \dots, \gamma_n} = \frac{N!}{\gamma_1! \dots \gamma_n!}$.)

Define the auxiliary polynomial

$$g_P(x_1, \dots, x_n, t) = \sum_{|\alpha|=d} c_\alpha \prod_{i=1}^n x_i(x_i - t) \cdot \dots \cdot (x_i - t(\alpha_i - 1)),$$

which is independent of N. Observe that $g_P(\mathbf{x}, 0) = P(\mathbf{x})$, and

$$\binom{N}{\beta_1, \dots, \beta_n} g_P(\beta_1, \dots, \beta_n, 1) = \text{coefficients of } x^\beta \text{ in } (x_1 + \dots + x_n)^N P(x).$$

To complete the proof of Theorem 1, the following claim will suffice.

Goal: Show that $g_P(\beta_1, \ldots, \beta_n, 1) > 0$ for all β with $|\beta| = d + N$, for $N \gg 0$.

By assumption, $g_P(\mathbf{x}, 0) = P(\mathbf{x}) > 0$ for all $x \in \mathbb{R}^n_{\geq 0} \setminus \mathbf{0}$. By continuity, there exist open neighborhoods $\mathbf{x} \in S_{\mathbf{x}} \subset \mathbb{R}^n$ and $0 \in U_{\mathbf{x}} \subset \mathbb{R}$ such that $g_P > 0$ on $S_{\mathbf{x}} \times U_{\mathbf{x}}$. Consider the unit simplex

$$\Delta_n = \{\mathbf{x} \in \mathbb{R}^n \mid x_i \geq 0, x_1 + \cdots + x_n = 1\}.$$

This has an open cover determined by $(S_{\mathbf{x}} \cap \Delta_n)_{\mathbf{x} \in \Delta_n}$. Use Heine-Borel to get a finite subcover $\{S_{\mathbf{x}} \cap \Delta_n\}_{\mathbf{x} \in X}$, where $X \subset \Delta_n$ is some finite set. Set

$$U = \bigcap_{\mathbf{x} \in X} U_{\mathbf{x}},$$

an open interval containing $t = 0$.
By homogeneity (and note that $\deg(g) = d$),

$$g_P(\beta_1, \ldots, \beta_n, 1) = (d+N)^d \, g_P\left(\frac{\beta_1}{d+N}, \ldots, \frac{\beta_n}{d+N}, \frac{1}{d+N}\right).$$

Observe:

1. $\left(\frac{\beta_1}{d+N}, \ldots, \frac{\beta_n}{d+N}\right) \in \Delta_n$.

2. $\frac{1}{d+N} \in U$ for $N \gg 0$.

Therefore, $g_P(\beta_1, \ldots, \beta_n, 1) > 0$, and the goal is achieved. □