MATH 8102 Fall 2026 — Scribe: Marc Abi Aad
August 26, 2026 — Editor: Tim Duff

Lecture 2: Farkas' Lemma

1 Farkas' Lemma

Given a polyhedron $P \subset \mathbb{R}^n$ and a linear cost function $f(\mathbf{x}) = \mathbf{c}^T\mathbf{x}$ ($\mathbf{x} \in \mathbb{R}^n, \mathbf{c} \in \mathbb{R}^n$ fixed), a *linear program* is an optimization problem having the form

$$\max f(\mathbf{x}) \text{ such that } \mathbf{x} \in P. \tag{1}$$

If $P = \emptyset$, the linear program (1) is *infeasible.*

Theorem 1 (Farkas' Lemma)**.** *For any polyhedron $P \subset \mathbb{R}^n$ with H-representation $A\mathbf{x} \leq b, A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m$, exactly one of two possibilities holds:*

(1) $P \neq \emptyset$,

(2) there exists $\mathbf{y} \in \mathbb{R}^m_{\geq 0} = \{\mathbf{y} \in \mathbb{R}^m \mid y^{(i)} \geq 0, \forall i\}$ such that $A^T\mathbf{y} = 0$ and $\mathbf{y}^T b < 0$.

Proof. We first check mutual exclusivity: suppose that $\mathbf{y}$ satisfying (2) exists and $\mathbf{x} \in P$, then $A\mathbf{x} \leq b$ hence $\mathbf{y}^T(A\mathbf{x}) \leq \mathbf{y}^T b$ (since $\mathbf{y} \geq 0$). This would imply that $(y^T A)\mathbf{x} = 0 < 0$, a contradiction.
It is now enough to show that (1) or (2) hold for any polyhedron P. We use induction on n:

- Base case $n = 1$: There exist $a_1, \ldots, a_{m'}, b_1, \ldots, b_m \in \mathbb{R}$ such that the H-representation of P is given by

$$A = \begin{pmatrix} -1 \\ \vdots \\ -1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \text{ and } b = \begin{pmatrix} -a_1 \\ \vdots \\ -a_{m'} \\ b_{m'+1} \\ \vdots \\ b_m \end{pmatrix}.$$

  Here, $P = \emptyset$ if and only if there exist i, k such that $a_i > b_k$, which is true if and only if $\mathbf{y} = e_i + e_{m'+k}$ satisfies $A^T\mathbf{y} = 0$ and $\mathbf{y}^T b = -a_i + b_k < 0$.

- Induction step $n > 1$: We will use Fourier-Motzkin Elimination. Let (A, b) be an H-representation of P and (A', b') an H-representation of $\pi(P) \subset \mathbb{R}^{n-1}$. Observe that $\pi(P) = \emptyset$ if and only if $P = \emptyset$, and assume that $P = \emptyset$. Then, by induction, there exists $\mathbf{y}' \in \mathbb{R}^m_{\geq 0}$ such that
$$(A')^T \mathbf{y}' = 0 \text{ and } (b')^T \mathbf{y}' < 0, \tag{2}$$
We may assume that every row of $\begin{pmatrix} A' & 0 & b' \end{pmatrix}$ is the sum of two rows from $\begin{pmatrix} A & b \end{pmatrix}$. Using $\mathbf{y}'$, we can write $-e_n \in \mathbb{R}^n$ as a nonnegative combination of rows of $\begin{pmatrix} A' & b' \end{pmatrix}$: given $\mathbf{y}'$ as in (2), replace $\mathbf{y}'$ with $\mathbf{y} = \frac{-1}{(b'^T \mathbf{y}')} \mathbf{y}' \in \mathbb{R}^m_{\geq 0}$. Equation (2) then becomes
$$(A')^T \mathbf{y} = 0 \text{ and } (b')^T \mathbf{y} = -1.$$
We may rewrite this as
$$-e_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} (A')^T \mathbf{y} \\ (b')^T \mathbf{y}) \end{pmatrix} = \begin{pmatrix} (A')^T \\ (b')^T \end{pmatrix} \mathbf{y}.$$
Thus, $-e_{n+1} \in \mathbb{R}^{n+1}$ is a nonnegative combination of rows of $\begin{pmatrix} A' & 0 & b' \end{pmatrix}$ hence a nonnegative combination of rows of $\begin{pmatrix} A & b \end{pmatrix}$, which implies (2).

$\square$

Remark 1. *The vector* $\mathbf{y}$ *appearing in Ferkas' Lemma is called a* dual infeasibility certificate.

Similar certificates exist in polynomial optimization. For example, the *real Nullstellensatz*, to be discussed later, gives a certificate when the feasible set is a real variety.