MATH 8102 Fall 2026 — August 24, 2026

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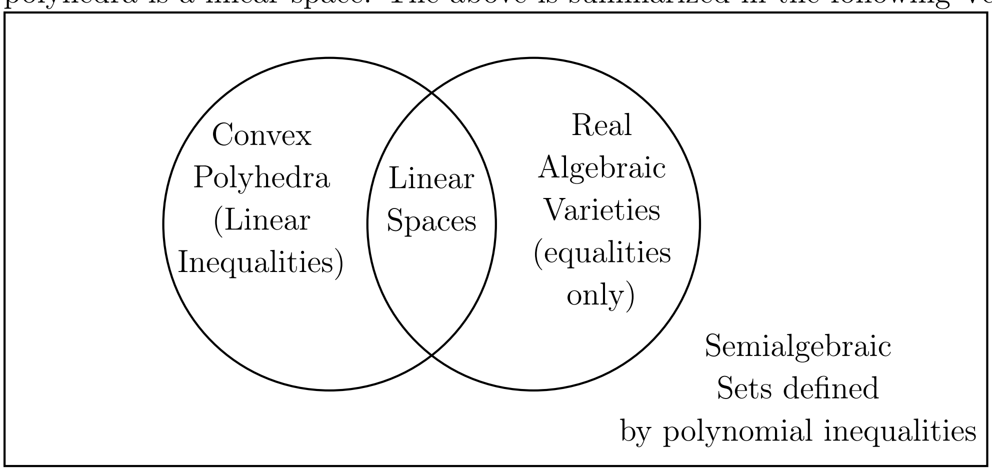
Lecture 1: Overview, Polyhedra, Projection of Polyhedra

1 Quick Overview

The field of real numbers $\mathbb{R}$ enjoys a natural order on its elements. Therefore, the study of *semialgebraic sets* (sets in $\mathbb{R}^n$ defined by polynomial inequalities) becomes relevant. In fact, semialgebraic sets generalize real algebraic varieties (sets in $\mathbb{R}^n$ defined by polynomial equalities). Indeed, the set defined by the equality $g(\mathbf{x}) = 0$ for some polynomial g can also be defined by two inequalities: $g(\mathbf{x}) \geq 0$ and $g(\mathbf{x}) \leq 0$. A special type of semialgebraic sets are *convex polyhedra*: sets in $\mathbb{R}^n$ defined by linear inequalities. A set in $\mathbb{R}^n$ that is both a real algebraic variety and a convex polyhedra is a linear space. The above is summarized in the following Venn diagram:



2 Polyhedra (Theobald Section 2.1)

Definition 1 (Polyhedron)**.** *A polyhedron $P \subset \mathbb{R}^n$ is the intersection of finitely-many closed half spaces. This means that for some $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$,*

$$P = \{x \in \mathbb{R}^n \mid Ax \leq b\}. \tag{1}$$

Equation (1) is called an H-representation of P.

Remark 1. *On $\mathbb{R}^m$, we define the partial order $\leq$ as follows: $y_1 \leq y_2$ for $y_1, y_2 \in \mathbb{R}^m$ if and only if $y_1^{(i)} \leq y_2^{(i)}$ for all $i = 1, \dots, m$.*

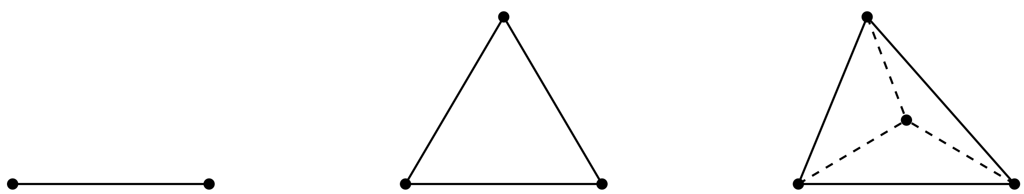
Remark 2. *Write* $A = \begin{pmatrix} a_1^T \\ \vdots \\ a_n^T \end{pmatrix}$ *and* $b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$*, then*

$$P = \bigcap_{i=1}^{m} \underbrace{\{x \mid a_i^T x \leq b_i\}}_{\text{a closed half space}}.$$

Exercise 1. *Show that any polyhedron* $P \subset \mathbb{R}^n$ *is closed (in the Euclidean topology) and convex (ie, if* $x, y \in P$*, then* $\lambda x + (1-\lambda)y \in P$ *for all* $\lambda \in [0,1]$*).*

Example 1. *The unit simplex in* $\mathbb{R}^n$ *(or the standard simplex) is the set*

$$\Delta_n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_i \geq 0 \text{ for } i = 1, \dots, n, \text{ and } x_1 + \cdots + x_n = 1\}.$$



Δ_2 Δ_3 Δ_4

In fact,

$$\Delta_n = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n \;\middle|\; \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & \ddots & \\ & & & -1 \\ 1 & 1 & \dots & 1 \\ -1 & -1 & \dots & -1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \leq \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ -1 \end{pmatrix} \right\}$$

Theorem 1 (Fourier-Motzkin Elimination Theorem (Theorem 2.1))**.** *For* $n \geq 2$*, let* $P \subset \mathbb{R}^n$ *be a polyhedron, and let* $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ *be the projection defined by* $\pi(x_1, \dots, x_n) = (x_1, \dots, x_{n-1})$*. For any polyhedron* $P \subset \mathbb{R}^n$*,* $\pi(P) \subset \mathbb{R}^{n-1}$ *is a polyhedron.*

Proof. Separate the H-representation (1) for P into 3 types of inequalities:

$$\sum_{j=1}^{n-1} a_{ij} x_j + x_n \leq b_i, 1 \leq i \leq m' \tag{2}$$

$$\sum_{j=1}^{n-1} a_{ij}x_j - x_n \leq b_i, m'+1 \leq i \leq m'' \tag{3}$$

$$\sum_{j=1}^{n-1} a_{ij}x_j \leq b_i, m''+1 \leq i \leq m. \tag{4}$$

Equations (2) and (3) hold if and only if

$$\max_{m'+1 \leq i \leq m''} \left(\sum a_{ij}x_j - b_i \right) \leq x_n \leq \min_{1 \leq k \leq m'} \left(b_k - \sum a_{ij}x_j \right).$$

Note that, given $(x_1, \ldots, x_{n-1}) \in \pi(P)$, there exists $(x_1, \ldots, x_n) \in \pi^{-1}(x_1, \ldots, x_{n-1}) \cap P$ if and only if the closed interval

$$\left[\max_{m'+1 \leq i \leq m''} \left(\sum a_{ij}x_j - b_i \right), \min_{1 \leq k \leq m'} \left(b_k - \sum a_{ij}x_j \right) \right]$$

is nonempty. We conclude that $\pi(P)$ is defined by equation (4) and the equations

$$\sum_{j=1}^{n-1} a_{ij}x_j - b_i \leq b_k - \sum_{j=1}^{n-1} a_{kj}x_j \text{ for } 1 \leq k \leq m', m'+1 \leq i \leq m'',$$

which shows that $\pi(P)$ is a polyhedron.

$\square$

Tarski's Theorem is the nonlinear analogue of Fourier-Motzkin elimination. A semialgebraic set $S \subset \mathbb{R}^n$ can be defined as any set of the form

$$S = \bigcup_{i=1}^{m} \left\{ x \in \mathbb{R}^n \mid g_{ij}(x) \geq 0, j = 1, \ldots, m_i, h_i(x) = 0 \right\},$$

where g_{ij} and h_i are polynomials.

Theorem 2 (Tarski's Theorem, or Tarski-Seidenberg Theorem)**.** *For $n \geq 2$, the projection $\pi(S) \subset \mathbb{R}^{n-1}$ of any semialgebraic set $S \subset \mathbb{R}^n$ is semialgebraic.*