

Lecture 18: Examples of Hermite Matrices, Relative Newton Sums

1 Hermite Matrix – Examples

Example 1. Let $f(t) = t^2 + at + b$. Then

$$H(f) = \begin{pmatrix} P_0(f) & P_1(f) \\ P_1(f) & P_2(f) \end{pmatrix}.$$

Using the Newton–Girard identities, we obtain

$$P_0(f) = \alpha_1^0 + \alpha_2^0 = 2,$$

$$P_1(f) = -a,$$

$$P_2(f) = a^2 - 2b.$$

Therefore,

$$H(f) = \begin{pmatrix} 2 & -a \\ -a & a^2 - 2b \end{pmatrix}.$$

Moreover, $\det(H(f)) = a^2 - 4b$. We can write

$$H(f) = \underbrace{\begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}}_L \underbrace{\begin{pmatrix} 2 & 0 \\ 0 & \frac{a^2 - 4b}{2} \end{pmatrix}}_D \underbrace{\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}}_{L^T}.$$

Consequently,

$$\operatorname{sgn}(H(f)) = \operatorname{sign}(2) + \operatorname{sign}\left(\frac{a^2 - 4b}{2}\right) = \begin{cases} 2, & a^2 - 4b > 0, \\ 1, & a^2 - 4b = 0, \\ 0, & a^2 - 4b < 0. \end{cases}$$

Example 2 (Degree 3 – Case (Depressed)). Let $f(t) = t^3 + at + b$. Then

$$H(f) = \begin{pmatrix} P_0(f) & P_1(f) & P_2(f) \\ P_1(f) & P_2(f) & P_3(f) \\ P_2(f) & P_3(f) & P_4(f) \end{pmatrix} = \begin{pmatrix} 3 & 0 & -2a \\ 0 & -2a & -3b \\ -2a & -3b & 2a^2 \end{pmatrix}.$$

We have the decomposition

$$H(f) = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ * & 1 & 0 \\ * & * & 1 \end{pmatrix}}_L \underbrace{\begin{pmatrix} 3 & 0 & 0 \\ 0 & \frac{-6a}{3} & 0 \\ 0 & 0 & \frac{-4a^3 - 27b^2}{-6a} \end{pmatrix}}_D \underbrace{\begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}}_{L^T}.$$

Principal minors: If $a \neq 0$,

- $\det(P_0(f)) = 3$.
- $\det \begin{pmatrix} P_0(f) & P_1(f) \\ P_1(f) & P_2(f) \end{pmatrix} = -6a$.
- $\det(H(f)) = -4a^3 - 27b^2$.

If $a \neq 0$, set $\Delta = -4a^3 - 27b^2$. Then

$$\text{sgn}(H(f)) = 1 - \text{sign}(a) + \text{sign}\left(\frac{4a^3 + 27b^2}{a}\right) = 1 - \text{sign}(a) (1 - \text{sign}(\Delta)) = \begin{cases} 3, & \Delta < 0, \\ 1, & \Delta > 0, \\ 2, & \Delta = 0. \end{cases}$$

Notice that $\Delta \leq 0 \Rightarrow a \leq 0$.

2 Relative Newton Sums

Let $f \in k[t]$ be monic of degree n , with roots $\alpha_1, \dots, \alpha_n \in \bar{k}$, and let $g \in k[t]$. We define the relative Newton sums as follows:

$$P_k(f; g) = \sum_{i=1}^n g(\alpha_i) \alpha_i^k.$$

Notice that $P_k(f; 1) = P_k(f)$. We define the generalized Hermite matrix

$$H(f; g) = \begin{pmatrix} P_0(f; g) & P_1(f; g) & \cdots & P_{n-1}(f; g) \\ P_1(f; g) & P_2(f; g) & \cdots & P_n(f; g) \\ \vdots & \vdots & \ddots & \vdots \\ P_{n-1}(f; g) & P_n(f; g) & \cdots & P_{2n-2}(f; g) \end{pmatrix}.$$

Theorem 1. a. If $k = \bar{k}$ is algebraically closed, then

$$\text{rank}(H(f; g)) = \#\{\text{distinct roots } \alpha_i \text{ of } f \mid g(\alpha_i) \neq 0\}.$$

b. If $k = \mathbb{R}$ is real closed, then

$$\text{sgn}(H(f; g)) = \#\{\text{distinct real roots of } f\}.$$

Proof. Proof on Monday. □

Example 3. If $g(t) = b_n t^n + b_{n-1} t^{n-1} + \dots + b_0$, then

$$P_k(f; g) = \sum_{i=1}^n g(\alpha_i) \alpha_i^k = \sum_{i=1}^n \sum_{\ell=0}^m b_\ell \alpha_i^{k+\ell} = \sum_{\ell=0}^m b_\ell P_{k+\ell}(f).$$

Let $f(t) = t^3 + at + c$ and $g(t) = t - c$. Then $P_k(f; g) = P_{k+1} - cP_k$. Hence, Then

$$H(f; g) = \begin{pmatrix} 3c & -2a & -2ac - 3b \\ -2a & -2ac - 3b & 2a^2 - 3bc \\ -2ac - 3b & 2a^2 - 3bc & 2a^2c + 5ab \end{pmatrix}.$$