

Lecture 17: Signature and Rank of the Hermite Matrix, Newton–Girard Formulas

1 Signature and Rank of the Hermite Matrix

Theorem 1. For f non-constant:

- a. $\text{rank}(H(f)) = \# \left\{ \text{distinct roots of } f, \text{ in an algebraic closure } \overline{\mathbb{K}} \supseteq \mathbb{K} \right\}.$
- b. If $\mathbb{K} = R$ real-closed, then $\text{sign}(H(f)) = \#\{\text{distinct roots of } f \text{ in } R\}.$

Proof. Finish proof of (b).

Notation: Let $\alpha_1, \dots, \alpha_{r'}$ be the distinct real roots of f . Additionally, $\alpha_{r'+1}, \dots, \alpha_r$ are the distinct non-real roots, ordered such that

$$\overline{\alpha_i} = \alpha_{i+s}, \quad i = r' + 1, \dots, \frac{r' + r}{2}$$

where $s = \#$ non-real conjugate pairs. We remind that

$$\vec{x}^T H(f) \vec{x} = (V(f)\vec{x})^T (V(f)\vec{x}).$$

which can be rewritten as follows :

$$\begin{aligned} \vec{x}^T H(f) \vec{x} &= \sum_{i=1}^r \text{ord}_{\alpha_i}(f) (L_{\alpha_i}(\vec{x}))^2 \\ &= \sum_{i=1}^{r'} \text{ord}_{\alpha_i}(f) (L_{\alpha_i}(\vec{x}))^2 + \sum_{i=r'+1}^{(r+r')/2} 2 \text{ord}_{\alpha_i}(f) [\text{Re } L_{\alpha_i}(\vec{x})^2 - \text{Im } L_{\alpha_i}(\vec{x})^2]. \end{aligned}$$

Let

$$L_{\alpha_i}(\vec{x}) = a_i^T \vec{x}, \quad i = 1, \dots, r'$$

$$\text{Re}(L_{\alpha_i})(\vec{x}) = b_i^T \vec{x}, \quad i = r' + 1, \dots, \frac{r + r'}{2},$$

$$\text{Im}(L_{\alpha_i})(\vec{x}) = c_i^T \vec{x}, \quad i = r' + 1, \dots, \frac{r + r'}{2}.$$

$a_i^T, b_i^T, c_i^T \in \mathbb{R}^n$. Since $\alpha_1, \dots, \alpha_r$ are distinct, the coefficient vectors of $L_{\alpha_1}, \dots, L_{\alpha_r}$ are linearly independent over $\mathbb{C}$. Hence $a_1^T, \dots, a_{r'}^T, b_1^T, \dots, c_i^T, \dots$ are $\mathbb{R}$ -linearly independent. So we can write

$$H(f) = L^T D L,$$

where

$$L = \begin{pmatrix} a_1^T \\ \vdots \\ a_{r'}^T \\ b_{r'+1}^T \\ \vdots \\ c_{r'+1}^T \\ \vdots \\ d_1^T \\ \vdots \\ d_{n-r}^T \end{pmatrix} \quad \text{complete } \{a_i, b_i, c_i\} \text{ to a basis of } \mathbb{R}^n.$$

and

$$D = \text{diag}(\text{ord}_{\alpha_1}(f), \dots, \text{ord}_{\alpha_{r'}}(f), 2 \text{ord}_{\alpha_{r'+1}}(f), \dots, -2 \text{ord}_{\alpha_{r'+1}}(f), \dots, 0, \dots, 0).$$

Notice that $\text{ord}_{\alpha_1}(f), \dots, \text{ord}_{\alpha_{r'}}(f)$ contribute r' to the signature; and

$$2 \text{ord}_{\alpha_{r'+1}}(f), \dots, -2 \text{ord}_{\alpha_{r'+1}}(f)$$

cancel in signature. □

2 Newton–Girard Identities

We recall that if $\mathbb{K}$ is a field, $f(t) = t^n + a_1 t^{n-1} + \dots + a_n \in \mathbb{K}[t]$ with roots $\alpha_1, \dots, \alpha_n \in \overline{\mathbb{K}}$, where $\overline{\mathbb{K}}$ is some algebraic closure of $\mathbb{K}$, then the **k -th power sum** (Newton sum) is

$$p_k(f) = \alpha_1^k + \alpha_2^k + \dots + \alpha_n^k$$

We show that

$$p_k(f) = -(a_1 p_{k-1}(f) + \dots + a_{k-1} p_1(f) + k a_k).$$

Proof. Recall the elementary symmetric polynomials $e_1(\vec{x}), \dots, e_m(\vec{x}) \in k[x_1, \dots, x_m]$,

$$e_k(\vec{x}) = \sum_{1 \leq i_1 < \dots < i_k \leq n} x_{i_1} x_{i_2} \dots x_{i_k}.$$

Vieta's formulas give :

$$e_k(\alpha_1, \dots, \alpha_n) = (-1)^k a_k, \quad \text{for } k = 1, \dots, n.$$

Work in the ring $S = \mathbb{K}[x_1, \dots, x_n][[t]]$ of formal power series. Consider

$$\prod_{i=1}^n (1 + tx_i) = \sum_{k=0}^n e_k(\vec{x}) t^k. \quad (1)$$

Taking the logarithmic derivative, we obtain ¹

$$\begin{aligned} \frac{\frac{d}{dt} \prod_{i=1}^n (1 + tx_i)}{\prod_{i=1}^n (1 + tx_i)} &= \sum_{i=1}^n \frac{x_i}{1 + tx_i} \\ &= \sum_{i=1}^n \sum_{j=0}^{\infty} (-1)^j x_i^{j+1} t^j, \quad k = j + 1 \\ &= \sum_{i=1}^n \sum_{k=1}^{\infty} (-1)^{k-1} x_i^k t^{k-1} \\ &= \sum_{k=1}^{\infty} (-1)^{k-1} \tilde{p}_k(\vec{x}) t^{k-1}. \end{aligned}$$

where $\tilde{p}_k(\vec{x}) = x_1^k + \dots + x_n^k$ Using 1 we obtain :

$$\frac{\sum_{k=1}^n k e_k(\vec{x}) t^{k-1}}{\sum_{k=0}^n e_k(\vec{x}) t^k} = \sum_{k=1}^{\infty} (-1)^{k-1} \tilde{p}_k(\vec{x}) t^{k-1}.$$

or equivalently,

$$\sum_{k=1}^n k e_k(\vec{x}) t^{k-1} = \left(\sum_{k=0}^n e_k(\vec{x}) t^k \right) \left(\sum_{k=1}^{\infty} (-1)^{k-1} \tilde{p}_k(\vec{x}) t^{k-1} \right) = \sum_{k=1}^{\infty} \left(\sum_{i=1}^k (-1)^{i-1} \tilde{p}_i(\vec{x}) e_{k-i}(\vec{x}) \right) t^{k-1}.$$

¹Note that $1 + tx_i$ is a unit in S , for all $i = 1, \dots, n$.

Equate the coefficients of t^{k-1} and evaluate for $\vec{x} = (\alpha_1, \dots, \alpha_n)$:

$$ke_k(\alpha_1, \dots, \alpha_n) = \sum_{i=1}^k (-1)^{i-1} e_{k-i}(\alpha_1, \dots, \alpha_n) p_i(f)$$

therefore

$$ka_k = -(p_1(f)a_{k-1} + \dots + p_{k-1}(f)a_1 + p_k(f)).$$

as we desire. □