

Lecture 16: Hermite Matrix, Signature and Rank of the Hermite Matrix

1 Hermite count

Definition 1. Let $\mathbb{K}$ be a field, $f(t) = t^n + a_1 t^{n-1} + \cdots + a_n \in \mathbb{K}[t]$ with roots $\alpha_1, \dots, \alpha_n \in \overline{\mathbb{K}}$, where $\overline{\mathbb{K}}$ is some algebraic closure of $\mathbb{K}$. The **k -th power sum** (Newton sum) is

$$P_k(f) = \alpha_1^k + \alpha_2^k + \cdots + \alpha_n^k$$

We note that $P_k(f)$ is fixed under $\sigma \in \text{Aut}(K(\alpha_1, \dots, \alpha_n)/K)$ so $P_k(f) \in \mathbb{K}$.

Remark 1. We notice that $P_1(f) = -a_1$. More generally, we have the *Newton–Girard* identities

$$P_k(f) = - (P_{k-1}(f)a_1 + P_{k-2}(f)a_2 + \cdots + P_1(f)a_{k-1} + ka_k).$$

For instance,

$$P_2(f) = - (P_1(f)a_1 + 2a_2) = a_1^2 - 2a_2.$$

2 Hermite matrix

We define

$$H(f) = \begin{pmatrix} P_0(f) & P_1(f) & P_2(f) & \cdots & P_{n-1}(f) \\ P_1(f) & P_2(f) & P_3(f) & \cdots & P_n(f) \\ P_2(f) & P_3(f) & P_4(f) & \cdots & P_{n+1}(f) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ P_{n-1}(f) & P_n(f) & P_{n+1}(f) & \cdots & P_{2n-2}(f) \end{pmatrix}.$$

We observe that $H(f) \in \text{Sym}_n(\mathbb{K})$. If $(\mathbb{K}, P)$ is an ordered field, there exists $L \in \text{GL}_n(\mathbb{K})$ and $d_1, \dots, d_n \in \mathbb{K}$ such that

$$H(f) = L \begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{pmatrix} L^T. \tag{1}$$

This representation is not unique, but

$$\text{sign}_P(H(f)) = \sum_{i=1}^n \text{sgn}_P(d_i). \quad (2)$$

is independent from the representation 1 by *Sylvester's Law of inertia*. **(HW2)**. For $\mathbb{K} = R$ real-closed the spectral theorem **(HW2)** implies

$$H(f) = Q \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} Q^T \quad (3)$$

where Q is an orthogonal matrix, i.e. $Q \cdot Q^T = \text{id}_{R^n}$, and $\lambda_1, \dots, \lambda_n \in R$ are the eigenvalues of $H(f)$. So since the relation 2 is independent of the representation of $H(f)$ we obtain :

$$\text{sign } H(f) = \sum_{i=1}^n \text{sign}(\lambda_i).$$

3 Signature and Rank of the Hermite Matrix

Theorem 1. For f non-constant:

- a. $\text{rank}(H(f)) = \# \left\{ \text{distinct roots of } f, \text{ in an algebraically closure } \overline{\mathbb{K}} \supseteq \mathbb{K} \right\}.$
- b. If $\mathbb{K} = R$ real-closed, then $\text{sign}(H(f)) = \#\{\text{distinct roots of } f \text{ in } R\}.$

Proof. a. We consider the Vandermonde matrix

$$V(f) = \begin{pmatrix} 1 & \alpha_1 & \alpha_1^2 & \cdots & \alpha_1^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \alpha_n & \alpha_n^2 & \cdots & \alpha_n^{n-1} \end{pmatrix}$$

We have $V(f)^T V(f) = H(f)$. For any $\vec{x} \in \overline{\mathbb{K}}^n$,

$$\vec{x}^T H(f) \vec{x} = (V(f) \vec{x})^T (V(f) \vec{x}) = \sum_{i=1}^n L_{\alpha_i}(\vec{x})^2,$$

where L_{α_i} are linear forms. Up to relabeling, assume $\alpha_1, \dots, \alpha_r$ are precisely the distinct roots of f . Therefore

$$\vec{x}^T H(f) \vec{x} = \sum_{i=1}^r c_i L_{\alpha_i}(\vec{x})^2 \quad (4)$$

for some $c_i \in \mathbb{Z}$. So one can rewrite the Hermite matrix as follows :

$$H(f) = \underbrace{\begin{pmatrix} 1 & \cdots & 1 \\ \alpha_1 & \cdots & \alpha_r \\ \vdots & & \vdots \\ \alpha_1^{n-1} & \cdots & \alpha_r^{n-1} \end{pmatrix}}_{n \times r} \underbrace{\begin{pmatrix} c_1 & c_1 \alpha_1 & \cdots \\ \vdots & \vdots & \\ c_r & c_r \alpha_r & \cdots \end{pmatrix}}_{r \times n}.$$

Therefore we have

$$\# \left\{ \text{distinct roots in } \overline{\mathbb{K}} \right\} = r \geq \text{rank} (H(f)).$$

To see equality holds, let $g(t) = \prod_{i=1}^r (t - \alpha_i)$. Observe that $H(g)$ is an $r \times r$ submatrix of $H(f)$ with

$$\det H(g) = (\det V(g))^2 \neq 0.$$

Hence $\text{rank}(H(f)) \geq r$. This concludes (a).

b. For $\mathbb{K} = R$ real closed and $z \in R(i)$ write :

$$\text{Re}(z) = \frac{z + \bar{z}}{2} \quad \text{and} \quad \text{Im}(z) = \frac{z - \bar{z}}{2i}$$

where $\overline{x + yi} = x - yi$. If $L_\alpha(\vec{x})$ and $L_{\bar{\alpha}}(\vec{x})$ appear in 4, $\alpha \neq \bar{\alpha}$, then

$$L_\alpha^2(\vec{x}) + L_{\bar{\alpha}}^2(\vec{x}) = 2 \text{Re}(L_\alpha)^2 - 2 \text{Im}(L_\alpha)^2.$$

One can verify this as follows :

$$L_\alpha^2 + L_{\bar{\alpha}}^2 = (\text{Re}(L_\alpha) + i \text{Im}(L_\alpha))^2 + (\text{Re}(L_\alpha) - i \text{Im}(L_\alpha))^2 = 2 \text{Re}(L_\alpha)^2 - 2 \text{Im}(L_\alpha)^2.$$

Thus we rewrite 4 as

$$\vec{x}^T H(f) \vec{x} = \sum_{\substack{\alpha \in R \\ f(\alpha)=0}} L_\alpha^2(\vec{x}) + 2 \sum_{\substack{\alpha \notin R \\ f(\alpha)=0}} [\text{Re}(L_\alpha)^2 - \text{Im}(L_\alpha)^2].$$

To be continued on Wednesday...

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