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Hilbert's 17th Problem and Schmüdgen's Theorem (Theobald §6.5 – 6.6)

We retain the notation of the previous lecture—R is real-closed.

Theorem 1 (Solution to Hilbert's 17th Problem)**.** *For any polynomial $f \in R[\mathbf{x}] = R[x_1, \dots x_n]$ that is globally nonnegative on R^n, we have $f \in \Sigma R(\mathbf{x})^2$—that is, f can be written as a sum of squared rational functions over R.*

Proof. By the Krivine-Stengle Positivstellensatz, $\exists H, G \in \Sigma R[\mathbf{x}]^2$ and $k \geq 0$ with

$$fG = f^{2k} + H.$$

We may assume $f \neq 0$, as otherwise the conclusion holds trivially. This assumption implies G cannot be identically zero. Hence

$$f = \frac{G(f^{2k} + H)}{G^2} \in \Sigma R(\mathbf{x})^2.$$

□

Naturally, one would like to be able to write any globally nonnegative polynomial as a sum of *polynomial squares*, i.e. as an element of $\Sigma R[\mathbf{x}]^2$. Hilbert understood in the 19th century that this was generally impossible. For integers $n, d, \geq 1$, let us define

$$R[\mathbf{x}]_d = R[x_1, \dots, x_n]_d = \{f \in R[x_1, \dots, x_n] \mid f \text{ is homogeneous of degree } d\} \cup \{0\}.$$

This is an R-vector space of dimension $\binom{n+d}{d}$. For $n, d \geq 2$ with d even, we shall be interested in comparing the following two subsets of $R[\mathbf{x}]_d$:

$$\mathcal{P}_{n,d} = \{p \in R[\mathbf{x}]_d \mid p(\xi) \geq 0 \ \forall \xi \in R^n\},$$

$$\Sigma_{n,d} = R[\mathbf{x}]_d \cap \Sigma R[\mathbf{x}]^2.$$

Theorem 2 (Hilbert's Classification, 1888)**.** *For $n, d \geq 2$ with d even, we have $\Sigma_{n,d} \subsetneq \mathcal{P}_{n,d}$ except in the following three cases:*

1. *$n = 2$ (homogeneized univariate polynomials of even degree)*
2. *$d = 2$ (quadratic forms)*
3. *$(n, d) = (3, 4)$ (ternary quartics)*

A complete proof will have to wait until next week.