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10/26/26 Editor: Tim Duff

Systems of Polynomial Inequalities (Theobald §6.4)

The notion of a preorder, which we encountered earlier in the setting of fields, extends naturally to the setting of rings. Throughout, we will be concerned with the ring of multivariate polynomials $R[\mathbf{x}] = R[x_1, \ldots, x_n]$ over a real-closed field R. A *preorder* $P \subset R[\mathbf{x}]$ is any set closed under addition and multiplication such that $a^2 P \subset P$ for all $a \in R[\mathbf{x}]$. As before, a preorder is *proper* if $-1 \notin P$, and we define the preorder generated by finitely-many polynomials $g_1, \ldots, g_m \in R[\mathbf{x}]$ as

$$\mathrm{PO}(g_1, \ldots, g_m) = \left\{ \sum_{\mathbf{e} \in \{0,1\}^m} a_{\mathbf{e}} g_1^{e_1} \cdots g_m^{e_m} \mid a_{\mathbf{e}} \in \Sigma R[\mathbf{x}]^2 \right\}. \tag{1}$$

This preorder can be used to decide emptiness of the *basic closed semialgebraic set*

$$\mathcal{S}(g_1, \ldots, g_m) = \{\xi \in R^n \mid g_i(\xi) \geq 0 \quad \forall i = 1, \ldots m\}. \tag{2}$$

This follows from the following *Positivstellensatz-lite.*

Theorem 1. *For $g_1, \ldots, g_m \in R[\mathbf{x}]$, R real-closed, TFAE:*

1. $-1 \in \mathrm{PO}(g_1, \ldots, g_m)$
2. $\mathcal{S}(g_1, \ldots, g_m) = \emptyset$

To prove this, we first need the following lemma.

Lemma 1. *If $P \subset R[\mathbf{x}]$ is a proper preorder, then there exists a proper preorder P' containing P such that $R[\mathbf{x}] = P' \cup -P'$ and $I := P' \cap -P'$ is a prime ideal in $R[\mathbf{x}]$.*

Proof. As in the case of fields, we construct P' as a maximal proper preorder containing P using Zorn's Lemma, and it suffices to check that P has the desired properties. The condition $R[\mathbf{x}] = P' \cup -P'$ also follows from the same proof. To see that $I = P \cap P'$ is a ideal, note first that

$$(P' + P' \subset P') \quad \wedge \quad (-P' + (-P') \subset -P') \quad \Rightarrow \quad I + I \subset I,$$

and $p \in I$ we have $-p \in I$, and hence for any $a \in R[\mathbf{x}]$

$$ap = (1/2)\left((a+1)^2 p + (a-1)^2(-p)\right) \in I.$$

Thus, I is an ideal. Finally, to see that I is prime, suppose $ab \in I$. We claim that either $a \in P'$ or $-b \in P'$. If this were not the case, then by the maximality of P' we would have $-1 \in P' + aP'$ and $-1 \in P' - bP'$, meaning there exist $a_1, a_2, b_1, b_2 \in P'$ such that

$$-1 = a_1 + aa_2 = b_1 - bb_2 \quad \Rightarrow \quad -aba_2b_2 = (a_1+1)(b_1+1) = r+1$$

for some $r \in P'$, and hence

$$-1 = r + aba_2b_2 \in P',$$

contradicting the properness of P'.

Thus, we proven the claim that $ab \in I$ implies $a \in P'$ or $b \in -P'$. A very similar proof proves that $ab \in I$ implies $a \in P'$ or $b \in P'$. Thus,

$$ab \in I \quad \Rightarrow \quad \left(a \in P'\right) \vee \left(b \in P \cap P' = I\right).$$

Copying the proof of our claim twice more proves the desired result

$$ab \in I \quad \Rightarrow \quad (a \in I) \vee (b \in I).$$

□

Proof of Theorem 1. (1) implies (2) is easy. For the converse, suppose $\mathrm{PO}(g_1, \ldots, g_m)$ is a proper preorder. Using the lemma, we extend to a maximal proper preorder Q with associated prime ideal $I = Q \cap -Q$. Consider the fraction field

$$\mathbb{F} = \mathrm{frac}\left(R[\mathbf{x}]/I\right).$$

We may give $\mathbb{F}$ an order $\leq_P$ by considering the extended preorder

$$Q^* = \left\{ \sum_{\mathbf{e} \in \{0,1\}^m} a_{\mathbf{e}} \overline{g}_1^{e_1} \cdots \overline{g}_m^{e_m} \mid a_{\mathbf{e}} \in \Sigma\mathbb{F}^2 \right\},$$

where $\overline{\bullet}$ denotes the residue mod-I. Note $\pm 1 \notin I$, so I cannot contain any nonzero elements of R; thus $\leq_P$ extends the unique order on R. Now we define

$$\xi = (\overline{x}_1, \ldots, \overline{x}_n) \in \mathbb{F}^n.$$

We have for each $j = 1, \ldots, m$ that

$$g_j(\mathbf{x}) \in \mathrm{PO}(g_1, \ldots, g_m) \subset Q \quad \Rightarrow \quad g(\xi) = \overline{g(\mathbf{x})} \in Q^* \subset P \quad \Rightarrow \quad g(\xi) \geq_P 0.$$

Thus, the first-order R-sentence

$$\exists x_1, \ldots, x_n \bigwedge_{j=1}^{n} \left(g_j(x_1, \ldots, x_n) \geq 0\right)$$

is satisfied in any real closure $\overline{\mathbb{F}} \supset \mathbb{F} \supset R$. Applying Tarski's transfer principle, this first-order sentence is also satisfied over R, i.e. $\mathcal{S}(g_1, \ldots, g_m) \neq \emptyset$. □