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Tarski's Transfer (Scheiderer §1.5 – §1.6)

Throughout R denotes a real-closed field.
Let $\vec{x} = (x_1, \ldots, x_n), \vec{y} = (y_1, \ldots, y_p), \xi \in R^n, f, g_1, \ldots, g_r \in \mathbb{Z}[\vec{x}, \vec{y}]$. Then there exists $F_i, Gij \in \mathbb{Z}[\vec{x}]$ such that T.F.A.E :

- $\exists \eta \in R^p$ with $(f(\xi, \eta) = 0) \wedge (\wedge_{\nu=1}^{r} g_\nu(\xi, \eta) > 0)$
- $\vee_{i=1}^{N} (F_i(\xi) = 0) \wedge (\wedge_{j=1}^{m_i} G_{ij}(\xi) > 0)$

Proof. $p > 1$ follows by induction,
$p = 1$ Use notation from Tarski's proof. Conditon (1) $\iff$ $\vee_{d=-1}^{m} \varphi_\alpha(\xi)$, where $0 \leq d \leq m$,

$$\varphi_d(\xi) = \left(\big(a_d(\xi) \neq 0\big) \bigwedge \big(\bigwedge_{k=d+1}^{n} a_k(\xi) = 0 \big) \right) \bigwedge \left(\bigvee_{\substack{\vec{s} \in \{-d, \ldots, d\}^{\{1,2\}^r} \\ \sum S_{\vec{e}} > 0}} \bigwedge_{\vec{e} \in \{1,2\}^r} \operatorname{sign} \big(\tilde{H_{d,e}}(\xi) = S_{\vec{e}} \big) \right)$$

Similarly for $d = -1$,

$$\varphi_{-1}(\xi) = \left[\bigwedge_{k=0}^{m} a_k(\xi) = 0 \right] \bigwedge \left[\bigwedge_{i=1}^{r} (g_r)_\xi \text{ is positive constant in } \vec{y} \right]$$

□

Corollary 1 (Tarski Transfer Principle (1st version))**.** *Let K be a real field, $R_i \supset K$ for $i = 1, 2$ are real closed fields containing K which induce the same order on K. For any $f_1, \ldots, f_r \in K[\vec{x}]$ and $\xi_i, \ldots, \xi_r \in \{0, \pm 1\}$, we have $\exists \xi \in R_1^n$ with $\operatorname{sign}(f_i(\xi)) = \xi_i \iff \xi \in R_2^n$ with $\operatorname{sign}(f_i(\xi)) = \xi_i$*

Proof. Without loss of generality, let $\xi_1 = 0, \xi_2 = \ldots = \xi_r = 1$. Introduce new variable $\vec{y} = (y_1, \ldots, y_N)$ so that $f_i(\vec{x}) = F_i(\vec{x}; \vec{b})$ for $F_1, \ldots, F_r \in \mathbb{Z}[\vec{x}, \vec{y}], \vec{b} \in K^N$. Apple the previous result, $\exists \xi \in R_i^n$ with prescribed signs iff

$$\bigvee_{i=1}^{N} \left(g_i(\vec{b}) \bigwedge \bigwedge_{j=1}^{m_i} h_{ij}(\vec{b}) > 0 \right) \text{ where } g_i, h_{ij} \in \mathbb{Z}[\vec{y}]$$

This condition depends only on the ordering of K, so it holds iff $\xi \in R_i^n$ for either i.
□

§1.6 Fix $\infty-$many variables $x_1, x_2, x_3, \ldots$, let A be a ring. An $\underline{A\text{-formula}}$ may be constructed as follows:

- For $f, g \in A[x_i | i \in S], S \subseteq \mathbb{N}$ finite, $(f < g)$ and $(f = g)$ are $A-$formulas.
- If φ and ψ are $A-$ formulas so are $\varphi \wedge \psi$ and $\neg\varphi$.
- If φ is $A-$formula and x a variable then $\forall x \in \varphi$ is an $A-$ formula.

This models 1-st order logic with constants in A. Use abbreviations as usual, $\exists\varphi = \neg(\forall\neg\varphi), f \neq g = \neg(f = g)$, etc.
If a variable x occurs in an $A-$ formula φ within a quantifier $\forall$ or $\exists$, we say x is $\underline{\text{bound}}$ in φ, otherwise x is $\underline{\text{free}}$ in φ.

$$Fr(\varphi) = \{x \in \{x_1, \ldots, \} \mid x \text{ is free in } \varphi\}$$

If $Fr(\varphi) = \emptyset$, we say φ is called an $\underline{A-\text{sentence}}$.
For φ any $A-$sentence φ, suppose $A \subset R$ is a subring of real closed field.
Write $R \models \varphi$ if $\exists a_1, a_2, \ldots \in R$ such that $\varphi|_{x_1=a_1, x_2=a_2, \ldots}$ is true. We say that R models φ.

Corollary 2 (Tarski Transfer Version 2)**.** *Let K be a real field, R_1 and R_2 real closed with $R_i \supset K$ for $i = 1, 2$ and inducing the same order on K. For any $K-$sentence φ,*

$$R_1 \models \varphi \iff R_2 \models \varphi$$