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Consequences of Tarski's Theorem

Throughout R denotes a real-closed field. Let $S \subseteq R^n, T \subseteq R^m$ be semialgebraic (R - semialgebraic). A map $f : S \longrightarrow T$ is semialgebraic if the graph

$$\mathrm{Gr}(f) = \{(s, f(s) \in R^{n+m} | s \in S\} \subseteq R^{n+1}$$

is semialgebraic.

Proposition 1. *Images and Pre-images of semialgebraic sets under semialgebraic maps are semialgebraic.*

Proof. Image : $f(s) = \pi_2(\mathrm{Gr}(F))$ is semialgebraic by Tarski.
Pre-image: If $T' \subset R^m$ semialgebraic then, $f^{-1}(T') = \pi_1\big(\mathrm{Gr}(f) \cap (R^n \times T')\big)$ semialgebraic (Boolean operations + Tarski).

$$\begin{array}{ccc} Gr(f) \subseteq R^{n+m} \ni (s_1, ..., s_n, t_1, .., t_m) & & \\ \pi_1 \big\downarrow & \searrow^{\pi_2} & \\ R^n \ni (s_1, ..., s_n) & & R^m \ni (t_1, .., t_m) \end{array}$$

□

Example 1. *If $S \subset \mathbb{R}^n$ semialgebraic then so are $\bar{S}$ (closure is Euclidean /order topology), $\partial S, \int(S)$ and* $\mathrm{conv}(S)$.
Recall : $\mathrm{conv}(S) = \{\xi \in \mathbb{R}^n | \exists k \geq 1, \xi_i, \ldots, \xi_k \in S, \lambda_1, \ldots, \lambda_k \in [0,1]$ *wiht* $\xi = \lambda_1\xi_1 + \ldots + \lambda_k\xi_k, 1 = \lambda_1 + \ldots + \lambda_k$.
We need a result from convex geometry.

Theorem 1 (Carathéodory)**.** *If $S \subseteq \mathbb{R}^n$ then any $\xi \in \mathrm{conv}(S)$ belongs to $\mathrm{conv}(\xi_1, \ldots, \xi_{n+1})$ for some $\xi_1, \ldots, \xi_{n+1} \in S$.*

Proof. Let $\xi \in \mathrm{conv}(S)$, so

$$\xi = \lambda_1\xi_1 + \ldots + \lambda_k\xi_k \text{ where } \xi_i \in S \text{ for all } i \in \{1, \ldots, k\}.$$

We are done unless $k \geq n+1$, If this holds, then $\xi_2 - \xi_1, \ldots, \xi_k - \xi_1$ are linearly dependent. So, there exists scalars, $c_2, \ldots c_k$, not all zeros, such that $\sum_{i=1}^{k-1} c_i(\xi_i - \xi_1) = 0$. Choose $\mu_1 = -c_2 - \ldots - c_k, \mu_i = c_i$ for all $i \in \{2, \ldots, k\}$ to get,

$$\sum_{i=1}^{k} \mu_i\xi_i = 0 \text{ and } \sum_{i=1}^{k} \mu_i = 0$$

then there exists some index i with $\mu_i > 0$. Among all such indices choose i such that $\alpha = \frac{\lambda_i}{\mu_i}$ is minimal.

Define for all $j = 1, \ldots, k$,

$$\tilde{\lambda_j} = \lambda_j - \alpha\mu_j, \text{ where by our choice of } \alpha, \tilde{\lambda_i} = 0.$$

Then consider,

$$\begin{aligned}\xi &= \lambda_1\xi_1 + \ldots + \lambda_k\xi_k = (\tilde{\lambda_1} + \alpha\mu_1)\xi_1 + \ldots + (\tilde{\lambda_k} + \alpha\mu_k)\xi_k \\ &= (\tilde{\lambda_1}\xi_1 + \ldots + \tilde{\lambda_k}\xi_k) + \alpha(\mu_1\xi_i + \ldots + \mu_k\xi_k) \\ &= (\tilde{\lambda_1}\xi_1 + \ldots + \tilde{\lambda_k}\xi_k) \in \text{conv}(\xi_1, \ldots, \hat{\xi_i}, \ldots, \xi_k) \text{ follows from note.}\end{aligned}$$

Note: $\sum_{j=1}^{k} \tilde{\lambda_j} = 1$ and $\tilde{\lambda_j} = \lambda_j - \mu_j\frac{\lambda_i}{\mu_i} \geq 0$ holds for $\mu_j > 0$ by definition of α. Therefore $\mu_j \leq 0$ then $\tilde{\lambda_j} \geq \lambda_j \geq 0$ hence $\tilde{\lambda_j} \in [0, 1]$ □

Corollary 1. *$S \subseteq \mathbb{R}^n$ semialgebraic, implies* $\text{conv}(S)$ *is semialgebraic.*

Proof. By Carathéodory's theorem, we have $\text{conv}(S) = \text{image}(f)$, where

$$\begin{aligned}f : S^{n+1} \times \Delta &\longrightarrow \mathbb{R}^n \\ (\xi_1, \ldots, \xi_{n+1}, \lambda_1, \ldots, \lambda_{n+1}) &\longrightarrow (\lambda_1\xi_1 + \ldots + \lambda_k\xi_k)\end{aligned}$$

Let $f_1, \ldots, f_n \in R[x_1, \ldots, x_n]$ be polynomial maps and consider consider a map,

$$\begin{aligned}R^n &\longrightarrow R^m \\ \vec{x} = \big(x_1, \ldots, x_n\big) &\longrightarrow \big(f_1(\vec{x}), \ldots, f_m(\vec{x})\big).\end{aligned}$$

This map is semialgebraic since $\text{Gr}(f) \subseteq R^{n+m}$ is algebraic. Therefore $S \subseteq \mathbb{R}^n$ semialgebraic implies $f(S) \subseteq \mathbb{R}^n$ is semialgebraic.

□

Note: $Z \subseteq R^n$ algebraic does not imply $f(Z)$ is algebraic.

Example 2. $Z = Z(xy - 1), f(y, y) = x$