

Lecture 15 Corollaries of Descartes Law and Hermite Counting

0.1 Descartes Law of Signs

$f = \sum_{i=0}^n a_i t^i \in R[t] \setminus 0$, R real closed. $N_+(f) = \sum_{\substack{a \in R \\ f(a)=0 \\ a>0}} \text{ord}_a(f)$. Descartes Rule of

Signs: $N_+(f) \leq \text{Var}(a_0, \dots, a_n)$.

Recap of the proof of Descartes Rule of Signs: Induction on $n = \deg(f)$ for $n > 1$. May assume $a_0 = f(0) > 0$. In general, $N_+(f) \leq N_+(f') + 1$ by Rolle's Theorem. If $N_+(f) = N_+(f')$ done by induction. Assume $N_+(f) = N_+(f') + 1$. Write $r \geq 1$ for the smallest index with $a_r \neq 0$.

Let ζ be the smallest positive zero of f . Then for any $\eta \in R$ with $0 < \eta < \zeta$ we must have $f'(\eta) \neq 0$ by Rolle's theorem. Also, $\text{sign}(f')$ should be constant on $(0, \zeta)$. Now, for $\eta \in (0, \zeta)$ sufficiently small, we have $\text{sgn}(a_r) = \text{sgn}(f'(\eta))$. (By the triangle inequality, enough to take $\min(1, \frac{r|a_r|}{(r+1)|a_{r+1}|}, \dots, \frac{r|a_r|}{n|a_n|})$. If we had $a_r > 0$, then f would be on $[0, \zeta]$, hence $0 < a_0 = f(0) \leq f(\zeta) = 0$.

Thus, $a_r < 0$. $\text{Var}(a_0, \dots, a_n) = \text{Var}(a_r, \dots, a_n) + 1 \leq N_+(f') + 1 = N_+(f)$. This completes the proof.

Corollary 1. *If $f \in R[t]$, R real-closed, which contains exactly m terms, then f has at most $2m - 2$ non-zero real roots(counted with multiplicity).*

Proof. $N_+(f) \leq m - 1$ and $N_+(f(-t)) \leq m - 1$.

□

Corollary 2. *Suppose $f = \sum_{i=0}^n a_i t^i \in R[t]$, R real-closed is real-rooted and $f \neq 0$. Then, the number of strictly positive roots(with multiplicity) of f is exactly $\text{Var}(a_0, \dots, a_n)$. Similarly, $N_+(f(-t)) = \text{Var}(a_0, -a_1, a_2, \dots, (-1)^n a_n)$.*

Proof. Assume $n = \deg f \geq 1$, $f(0) \neq 0$. Let $p = N_+(f)$, $p' = N_+(f(t))$. For any $x \in R$ define $W(x) = \text{Var}(a_0 + x, \dots, a_n + x)$. $W'(x) = \text{Var}(a_0 + x, -a_1 + x, \dots, (-1)^n(a_n + x))$. By Descartes Rule, $W(0) \geq p$, $W'(0) \geq p'$. Choose x sufficiently small so that $x \in (0, \min_{a_i \neq 0} |a_i|)$.

Observation 1 $x < \min_{a_i \neq 0} |a_i|$, every index where $W(0)$ changes sign must also give a sign change in $W(x)$. Similarly, any index of a sign change in $W'(0)$ gives a sign change

in $W'(x)$. Thus $[p \leq W(0) \leq W(x), p' \leq W'(0) \leq W'(x), p + p' = n, W(x) + W'(x) = n]$. If $p < W(0)$ or $p' < W'(0)$ then $n = p + p' < W(0) + W'(0) \leq W(x) + W'(x) = n$.

Observation 2 For $x \in (0, \min_{a_i \neq 0} |a_i|)$, each index $1, \dots, n$ gives exactly one sign change in either $W(x)$ or $W'(x)$, $p = W(0) = \text{Var}(a_0, \dots, a_n)$.

□

0.2 Hermite Counting

Goal: given $f \in R[t]$, R real-closed, determine the number of *distinct* roots of f in R .

Two approaches:

(1) Sturm sequences $\implies$ count on $[a, b] \subset R$.

(2) Hermite counting = based on linear algebra, generalized version incorporating sign conditions

Hermite Matrix For $f = t^n + a_1 t^{n-1} + \dots + a_n \in R[t]$, R real-closed. Let $\alpha_1, \dots, \alpha_n \in R(i)$ be the roots of f . For $k = 0, 1, 2, \dots$, define the k -th power sum $P_k(f) = \alpha_1^k + \dots + \alpha_n^k \in R$ by Galois Theory.