MATH 8102 Fall 2026 Scribe: Shanea Sebastian
10/19/26 Editor: Tim Duff

Tarski's Theorem

Throughout R denotes a real-closed field.

Lemma 1. *Let $g_1, \ldots, g_r \in R[t]$. Define $g := g_1 \ldots g_r$ and $h = g'(1-g^2)$. If there exists $\xi \in R$ with $g_i(\xi) > 0$ for all i, then there exists such a ξ further satisfying $h(\xi) = 0$.*

Proof. If g is constant then it is a trivial case. Suppose that $c_1 < c_2 < \ldots < c_N$ are the real roots of g.
If $N = 0$, then $\deg(g)$ is even and g_i are strictly positive on R by I.V.T. Then $\deg(g')$ is odd. Let ξ be any root of g' finishes the case.
Assuming $N > 0$, note g' has a root in each of the $N-1$ intervals $(c_1, c_2), \ldots, (c_{N-1}, c_N)$ (by Rolle's theorem). For $|t|$ sufficiently large, we have $1 - g(t)^2 < 0$. Since

$$1 - g(c_1)^2 = 1 - g(c_N)^2 = 1 > 0,$$

$1 - g^2$ has a root in each of the intervals $(-\infty, c_1)$ and (c_N, ∞). Therefore h has roots in $N + 1 \geq 1$ disjoint intervals covering $R \setminus \{c_1, \ldots, c_N\}$. Choose ξ a root of h on intervals where all g_i are positive.

□

Proof of Tarski's Theorem. Setup : Let $\varphi : A \longrightarrow R, S \subset R^{n+1}$, A-semialgebraic, $\pi : R^{n+1} \longrightarrow R$ given by $(t, x_1, \ldots, x_n) \longrightarrow (x_1, \ldots, x_n)$. We need to show $\pi(S) \subset R^n$ is A-semialgebraic.
Using normal form and

$$\pi(\bigcup_{i=1}^{m} S_i) = \bigcup_{i=1}^{m} \pi(S_i),$$

we may assume,

$$S = \{(t, \vec{x}) \in R^{n+1} | f(t, \vec{x}) = 0, g_j(t, \vec{x}) > 0, j = 1, \ldots, r\}$$

Write

$$f = \sum_{i=0}^{m} a_i(\vec{x}) t^i, \qquad g_j = \sum_{k=0}^{m_j} b_{j,k}(\vec{x}) t^k, \qquad m, m_1, \ldots, m_r \in \mathbb{Z}_{>0}, \qquad a_i, b_{j,k} \in \varphi(A)[\vec{x}].$$

For $\xi \in R^n$, write $h_\xi(t) := h(t, \xi)$.
Define for $0 \leq d \leq m = \deg(f)$,

$$\Sigma_d = \{\xi \in R^n | \deg(f_\xi) = d\}, \qquad \Sigma_{-1} = \{\xi \in R^n | f_\xi = 0\}$$

Enough to show $\pi(S) \cap \Sigma_d$ is A-semialgebraic for all $-1 \leq d \leq m$.

Case 1: $d = 0$, f_ξ is a nonzero contant, so $\pi(S) \cap \Sigma_0 = \phi$,

Case 2: $1 \leq d \leq m$, write, $f_\xi(t) = a_d(\xi)t^d + \ldots + a_0(\xi), \quad a_d(\xi) \neq 0,$

$$\begin{aligned}\pi(S) \cap \Sigma_d &= \{\xi \in R^n | \exists t \in R \text{ with } f_\xi(t) = 0 \text{ where } \deg(f_\xi) = d \text{ and } (g_i)_\xi(t) > 0\} \\ &= \{\xi \in R^n | \sum_{\vec{e} \in \{1,2\}^r} \operatorname{sign}\left(H\left(\frac{1}{a_d(\xi)} f_\xi, g_\xi^{\vec{e}}\right)\right) > 0\}, \qquad \text{where } g^{\vec{e}} = g_1^{e_1} . \cdots . g_r^{e_r}\end{aligned}$$

Define,

$$\widetilde{H}_{d,\vec{e}}(\xi) = (a_d(\xi))^{2m} H\left(\frac{1}{a_d(\xi)} f_\xi, g_\xi^{\vec{e}}\right)$$

whose entries are polynomial in $a_{i(\xi)}$ and $g_{i(\xi)}$ (Newton-Girard).
By generalized Hermite counting,

$$\pi(S) \cap \Sigma_d = \bigcup_{\vec{S} \in \{0, \pm 1 \ldots, \pm d\}^{2^r}} T(\vec{S}),$$

where for $\vec{S} = (S\vec{e})_{\vec{e} \in \{1,2\}^r}$, we define $T(\vec{S}) = \phi$ unless $\sum_{\vec{e} \in \{1,2\}^r} S\vec{e} > 0$, in which case,

$$T(\vec{S}) = \bigcap_{\vec{e} \in \{1,2\}^r} \{\xi \in R^n | \operatorname{sign}(\widetilde{H}_{d,\vec{e}}(\xi) = \vec{S}\vec{e}\},$$

$$S = \{(t, \vec{x}) \in R^{n+1} | f(t, \vec{x}) = 0, \ g_j(t, \vec{x}) > 0, \ j = 1, \ldots, r\}$$

Since $T(\vec{S}')$ is an intersection of sets defined by fixing the signatures of matrices of polynomials, it is A semialgebraic. Thus $\pi(S) \cap \Sigma_d$ is A-semialgebraic.

Case 3: d= -1,

$$\pi(S) \cap \Sigma_{-1} = \pi(S') \cap \Sigma_{-1} \text{ and } S = \{(t, \xi) \in R^{n+1} | h(t, \xi) = 0, \quad g_j(t, \xi) > 0 \ \ \forall j\},$$

with h as in Lemma, observe: $h_\xi \equiv 0 \iff g_\xi$ constant $\iff$ all $(g_j)_\xi$ constant.
Therefore,

$$\pi(S) \cap \Sigma_{-1} = \bigcap_{i=1}^{r} \{\xi \mid (g_j)_\xi > 0, \text{ and constant}\} \text{ is } A \text{ - semialgebraic.}$$

□