

Semi-Algebraic Set forms and Examples

Throughout R denotes a real-closed field.

Lemma 1. *A subset $M \subseteq R^n$ is A -semialgebraic if and only if M is of the form*

$$M = \bigcup_{i=1}^m (\mathcal{Z}(f_i) \cup \mathcal{U}(g_{i1}, \dots, g_{ir_i})) \quad (1)$$

Proof. First note that

$$\mathcal{S}(f_1, \dots, f_m) = \bigcap_{i=1}^m \mathcal{U}(-f_i)^c$$

is A -semialgebraic and so is

$$\mathcal{Z}(f_1, \dots, f_m) = \mathcal{S}(f_1, \dots, f_m) \cap \mathcal{S}(-f_1, \dots, -f_m).$$

Moreover,

$$\mathcal{Z}(f_1, \dots, f_m) = \mathcal{Z}(f_1^2 + \dots + f_m^2).$$

Now, observe if M is of the form in equation (1), then M is semialgebraic by the previous observations. For the converse, it suffices to show that sets of this form are closed under boolean combinations.

It is clear that sets of form (1) are closed under unions. Now to see intersections, note

$$(\mathcal{Z}(f) \cap \mathcal{U}(g_{11}, \dots, g_{1r_1})) \cap (\mathcal{Z}(f_2) \cap \mathcal{U}(g_{21}, \dots, g_{2r_2}))$$

and

$$\mathcal{Z}(f_1^2 + f_2^2) \cap \mathcal{U}(g_{11}, \dots, g_{1r_1}, g_{21}, \dots, g_{2r_2})$$

are the same set, giving us closed under intersections.

Finally, to see that sets of this form are closed under intersections, it suffices to show $(\mathcal{Z}(f) \cap (g_1, \dots, g_m))^c$ is of form (1). But

$$(\mathcal{Z}(f) \cap \mathcal{U}(g_1, \dots, g_m))^c = \mathcal{Z}(f)^c \cup \bigcup_{i=1}^n \mathcal{U}(g_i)^c$$

reducing the problem to showing $\mathcal{Z}(f)^c$ and $\mathcal{U}(g)^c$ is of form (1). Now, we have

$$\mathcal{Z}(f)^c = \mathcal{U}(f) \cup \mathcal{U}(-f)$$

and

$$\mathcal{U}(g)^c = \mathcal{U}(-g) \cup \mathcal{Z}(g)$$

finishing the proof. □

Lemma 2. Let $n \in \mathbb{Z}_{\geq 0}, k \in \mathbb{Z}$. Then the set

$$\{M \in \text{Sym}_n(R) : \text{sign}(M) = K\}$$

is $\mathbb{Z}$ -semialgebraic.

Proof. We will show, for fixed n and k , there are finitely many f_μ and $g_{\mu\nu} \in \mathbb{Z}[x_{ij} : 1 \leq i \leq j \leq n]$ such that $\text{sign}(M) = K$ if and only if

$$\bigvee_{\mu=1}^r (f_\mu(\mathbf{x}) = 0 \wedge (\bigwedge_{\nu=1}^{s_\mu} g_{\mu\nu}(\mathbf{x}) > 0)).$$

Consider the characteristic polynomial of $M = \begin{pmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ x_{12} & \ddots & \dots & \vdots \\ \vdots & \dots & \ddots & \vdots \\ x_{1n} & x_{2n} & \dots & x_{nn} \end{pmatrix}$. This is a polynomial of the form

$$\chi_M(t) = t^n + a_1(\mathbf{x})t^{n-1} + \dots + a_n(\mathbf{x})$$

where $a_i(\mathbf{x}) \in \mathbb{Z}[\mathbf{x}]$ where $a_1(\mathbf{x}) = -\text{Tr}(M)$ and higher order formulas are given in terms of principal minors.

By spectral theorem, $\chi_M(t)$ is real rooted. Moreover,

$$\text{sign}(M) = \text{var}(1, a_1(\mathbf{x}), \dots, a_n(\mathbf{x})) - \text{var}(1, -a_1(\mathbf{x}), a_2(\mathbf{x}), \dots, (-1)^n a_n(\mathbf{x})).$$

Then we win by taking each of $f_\mu, g_{\mu\nu}$ as $\pm a_i(\mathbf{x})$. □

Example 1. Consider an arbitrary 2×2 matrix

$$M = \begin{pmatrix} a & c \\ c & b \end{pmatrix}.$$

Then we have

$$\text{sign}(M) = \begin{cases} 2 & \text{if } a > 0 \wedge ab - c^2 > 0 \\ 1 & \text{if } ab - c^2 = 0 \wedge (a > 0 \vee b > 0) \\ 0 & \text{if } ab - c^2 < 0 \vee (a = b = c = 0) \\ -1 & \text{if } ab - c^2 = 0 \wedge (a < 0 \vee b < 0) \\ -2 & \text{if } a < 0 \wedge ab - c^2 < 0 \end{cases}$$

Lemma 3. Let $g_1, \dots, g_r \in R[t]$. Define $g := g_1 \dots g_r$ and $h = g'(1 - g^2)$. If there exists $\xi \in R$ with $g_i(\xi) > 0$ for all i , then $h(\xi) = 0$.

Proof. This is trivial if g is constant. Thus, assume not. Suppose $c_1 < \cdots < c_n$ are the real roots of g . If $N = 0$ then $\deg(g)$ is even and each g_i is strictly positive on R by Intermediate Value Theorem.

Now since $\deg(g)$ is even, we have $\deg(g')$ is odd and has at least one real root. Let ξ be one such root to finish this case. $\square$