

Semi-Algebraic Sets

Throughout R denotes a real-closed field. Set

$$f = t^n + a_1 t^{n-1} + \cdots + a_n$$

and

$$g = b_0 t^m + \cdots + b_m$$

The first part of class was used to fix Theorem 1 from the pervious class. To finish our studies of Hermite matrices, we prove the following lemma.

Lemma 1. *For all $k, m \in \mathbb{Z}_{\geq 0}$ and $n \in \mathbb{Z}_{\geq 1}$ there exists a universal polynomial*

$$P_k(x_1, \dots, x_n, y_0, \dots, y_m) \in \mathbb{Z}[x_1, \dots, x_n, y_0, \dots, y_m]$$

such that

$$P(a_1, \dots, a_n, b_0, \dots, b_m) = P_k(f, g)$$

for f and g as set above.

Proof. We define P_k recursively. Start by defining a function

$$\tilde{P}_l(x_1, \dots, x + m) = \begin{cases} n & l = 0 \\ -(x_1 \tilde{P}_{l-1}(\mathbf{x}) + \cdots + x_{l-1} \tilde{P}_1(\mathbf{x}) + l x_l) & l > 0 \end{cases}$$

Using this new function, we define

$$P_l(\mathbf{x}, \mathbf{y}) = \begin{cases} \sum_{s=0}^m y_s \tilde{P}_s(\mathbf{x}) & l = 0 \\ \sum_{s=0}^m y_s P_{l+s}(\mathbf{x}) & l > 0 \end{cases}$$

This function will work by Newton-Girard identities. □

1 Semi-Algebraic sets & Tarski's Theorem

Set up: R denotes a real closed field and $\varphi : A \rightarrow R$ is a ring homomorphism (A commutative with identity and φ is such that $\varphi(1_A) = 1_R$). Given $f_1, \dots, f_m \in R[x_1, \dots, x_n]$, we define

$$\mathcal{U}(f_1, \dots, f_m) := \{\xi \in R^n : f_i(\xi) > 0 \text{ for all } i\}$$

$$\mathcal{S}(f_1, \dots, f_m) := \{\xi \in R^n : f_i(\xi) \geq 0 \text{ for all } i\}$$

$$\mathcal{Z}(f_1, \dots, f_m) := \{\xi \in R^n : f_i(\xi) = 0 \text{ for all } i\}$$

Remark 1. We can define these sets for $f_1, \dots, f_m \in A[\mathbf{x}]$ via φ

Definition 1. A finite boolean combinations of sets having the form

$$\mathcal{U}(f_1, \dots, f_m)$$

for $f_1, \dots, f_m \in A[x]$ is said to be A -Semialgebraic

Theorem 1 (Tarski's Theorem). Let $A \subset R$. Consider the coordinate projection

$$\begin{aligned}\pi_n : R^m \times R^n &\rightarrow R^n \\ (\xi, \eta) &\mapsto \eta\end{aligned}$$

If $S \subset R^m \times R^n$ is an A -semialgebraic set, then $\pi(S) \subset R^n$ is also semialgebraic

Remark 2. This should remind the reader of the statement from Fourier-Motzkin elimination