

General Hermite Matrices and Polynomial Inequalities

Through out let R denote a real closed field. Let $f, g \in k[t]$ with

$$\begin{aligned} f(t) &= t^n + a_1 t^{n-1} + \cdots + a_n \\ g(t) &= b_0 t^m + b_1 t^{m-1} + \cdots + b_m \end{aligned}$$

Let $\alpha_1 \dots \alpha_n$ denote the roots of f . We define

$$\begin{aligned} P_k(f, g) &:= \sum_{j=1}^n g(\alpha_j) \alpha_j^k \\ &= \sum_{l=1}^m b_l P_{k+1}(f) \end{aligned}$$

We thus define a generalized Hermite matrix of f with respect to g as

$$H(f, g) := \begin{pmatrix} P_0(f, g) & P_1(f, g) & P_2(f, g) & \cdots \\ P_1(f, g) & P_2(f, g) & \cdots & \cdots \\ P_2(f, g) & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

Remark 1. Notice, $H(f, 1) = H(f)$ as defined before.

Theorem 1. a) $\text{rank}(H(f, g)) = \#\{\alpha_i : g(\alpha_i) \neq 0\}$

b) If $k = R$ then

$$\text{sign}(H(f, g)) = \sum \text{sign}(g(\alpha_i))$$

Proof. Part a's proof will be similar to when $g = 1$. Indeed, we have

$$H(f, g) = V(f)^T \begin{pmatrix} g(\alpha_1) & & & \\ & g(\alpha_2) & & \\ & & \ddots & \\ & & & g(\alpha_n) \end{pmatrix} V(f).$$

Thus we get

$$\mathbf{x} H(f, g) \mathbf{x} = \sum_{i=1}^n g(\alpha_i) L_{\alpha_i}(\mathbf{x})^2.$$

Then, using an LDL decomposition, we have

$$H(f, g) = L^T \begin{pmatrix} A & & \\ & B & \\ & & C \end{pmatrix} L$$

where the middle matrix is a block matrix such that A corresponds to real roots of f , $B = -C$ corresponds to the remaining complex roots of f . Then taking the signature gives the desired results.

For part b, if α_0 is a non-zero root of f , then

$$L_\alpha(\mathbf{x})^2 + L_{\bar{\alpha}}(\mathbf{x})^2$$

is a difference of squares of independent linear forms of $\mathbf{x}$ as before. It is sufficient to show that a weighted sum

$$\beta L_\alpha(\mathbf{x})^2 + \bar{\beta} L_{\bar{\alpha}}(\mathbf{x})^2$$

is also a difference of squares of independent linear forms of $\mathbf{x}$.

To show this, we write

$$\begin{aligned} L_\alpha(\mathbf{x}) &= a(\mathbf{x}) + i(\mathbf{x}), a, b \in \mathbb{R}[x] \\ \beta &= c + di, c, d \in R \end{aligned}$$

with a, b being $\mathbb{R}$ linearly independent. This yields an equation

$$\beta L_\alpha(\mathbf{x})^2 + \bar{\beta} L_{\bar{\alpha}}(\mathbf{x})^2 = 2c(a(\mathbf{x})^2 - b(\mathbf{x})^2) - 4d \cdot a(\mathbf{x})b(\mathbf{x}). \quad (1)$$

We now proceed by cases.

(Case 1: $d = 0$) Our second term on the right side of equation (1) vanishes, giving

$$\begin{aligned} \beta L_\alpha(\mathbf{x})^2 + \bar{\beta} L_{\bar{\alpha}}(\mathbf{x})^2 &= 2c(a(\mathbf{x})^2 - b(\mathbf{x})^2) \\ &= 2\sqrt{\pm c} \cdot a(\mathbf{x}) \mp \sqrt{\mp c} \cdot b(\mathbf{x}) \end{aligned}$$

(Case 2: $d \neq 0, c = 0$) Our first term on the right side of equation (1) vanishes, giving

$$\begin{aligned} \beta L_\alpha(\mathbf{x})^2 + \bar{\beta} L_{\bar{\alpha}}(\mathbf{x})^2 &= 4d \cdot a(\mathbf{x})b(\mathbf{x}) \\ &= (a(\mathbf{x}) - d \cdot b(\mathbf{x}))^2 - (a(\mathbf{x}) + d \cdot b(\mathbf{x}))^2 \end{aligned}$$

(Case 3: $dc \neq 0$) After some division by c , we get

$$\beta L_\alpha(\mathbf{x})^2 + \bar{\beta} L_{\bar{\alpha}}(\mathbf{x})^2 = 2c(a(\mathbf{x}) - \frac{d}{c}b(\mathbf{x}))^2 - \frac{2(c^2 + d^2)}{c}b(\mathbf{x})^2.$$

Since $\text{sign}(c) = \pm 1$, each gives a difference of squares of independent forms similar to case 1. $\square$

Remark 2. Notice, any finite number of side conditions can be enforced by taking the product of the condition polynomials.

0.1 Polynomial inequalities and Hermite Matrices

Fix $f, g_1, \dots, g_r \in R[t]$, $f \neq 0$. Let $\mathbf{e} \in \{1, 2\}^r$. Define

$$g^{\mathbf{e}} := \prod_{i=1}^r g_i^{e_i}$$

and

$$N_{\mathbf{e}} = \sum_{c \in R, f(c)=0} \text{sign}(g^{\mathbf{e}}(c))$$

Proposition 1. *The number of real roots of f such that $g_i(c) > 0$ for all $1 \leq i \leq r$ is*

$$2^{-r} \sum_{\mathbf{e} \in \{1, 2\}^r} N_{\mathbf{e}}$$

Proof. First observe for any choice $y_1, \dots, y_r \in \{1, -1, 0\}$ we have

$$\prod_{i=1}^r (y_i + y_i^2) = \begin{cases} 2^r & \text{if } y_i = 1 \text{ for all } i \\ 0 & \text{else} \end{cases}$$

Now

$$\begin{aligned} \sum_{\mathbf{e} \in \{1, 2\}^r} N_{\mathbf{e}} &= \sum_{\mathbf{e} \in \{1, 2\}^r} \sum_{c \in R: f(c)=0} \text{sign}(g_1(c)^{e_1} \cdots g_r(c)^{e_r}) \\ &= \sum_{c \in R: f(c)=0} \sum_{\mathbf{e} \in \{1, 2\}^r} \text{sign}(g_1(c)^{e_1} \cdots g_r(c)^{e_r}) \\ &= \sum_{c \in R: f(c)=0} \prod_{i=1}^r (\text{sign}(g_i(c) + g_i(c)^2)) \end{aligned}$$

with the last equality being evident because sign is a homomorphism. Applying the observation yields

$$\sum_{\mathbf{e} \in \{1, 2\}^r} N_{\mathbf{e}} = \sum_{c: f(c)=0, g_i(c) > 0} 2^r$$

and thus we win. □