Real AG HW 3 Due November 18.

1. In this exercise, you will determine the convex hull of the *special orthogonal group*, $\mathrm{SO}(3) = \{R \in \mathbb{R}^{3\times 3} \mid RR^T = \mathrm{id}_{\mathbb{R}^3},\ \det(R) = 1\}$.

   (a) Consider the polynomial map $Q : \mathbb{S}^3 \to \mathbb{R}^{3\times 3} \cong \mathbb{R}^9$ defined by

   $$Q(a,b,c,d) = \begin{bmatrix} a^2+b^2-c^2-d^2 & 2(bc-ad) & 2(bd+ac) \\ 2(bc+ad) & a^2+c^2-b^2-d^2 & 2(cd-ab) \\ 2(bd-ac) & 2(cd+ab) & a^2+d^2-b^2-c^2 \end{bmatrix},$$

   where $\mathbb{S}^3$ is the 3-sphere $\mathbb{S}^3 = \{(a,b,c,d) \in \mathbb{R}^4 \mid a^2+b^2+c^2+d^2 = 1\}$. Show that Q maps $\mathbb{S}^3$ two-to-one onto $\mathrm{SO}(3)$, and

   $$Q(a,b,c,d) = L\left(\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}\begin{bmatrix} a & b & c & d \end{bmatrix}\right),$$

   where $L : \mathbb{A}^9 \to \mathbb{R}^{3\times 3}$ is an invertible affine-linear map on the domain

   $$\mathbb{A}^9 = \{U \in \mathrm{Sym}_4(\mathbb{R}) \mid \mathrm{tr}(U) = 1\}.$$

   (Hint: to show Q is surjective and 2:1, solve for a^2, b^2, c^2, d^2 by combining diagonal entries, with at least one combination positive.)

   (b) Deduce that $X \in \mathrm{conv}(\mathrm{SO}(3))$ iff $L^{-1}(X)$ is *positive semidefinite*.

2. Use Tarski's theorem to show that the closure, in the Euclidean topology, of a semialgebraic set $S \subset \mathbb{R}^n$ is semialgebraic. What is the closure of

   $$S = \{(x,y) \in \mathbb{R}^2 \mid x^3 - x^2 - y^2 > 0\}?$$

3. Consider the family of univariate polynomials $f(t) = t^5 + at^3 + bt^2 + 1$. Let $\Delta(a,b) = \det H(f)$ denote the discriminant.

   (a) Use Descartes' rule of signs to determine the permissible numbers of positive and negative roots of f for all $(a,b) \in \mathbb{R}^2 \setminus \mathcal{Z}(\Delta)$.

   (b) Try to plot the connected components of $\mathbb{R}^2 \setminus \mathcal{Z}(\Delta)$ with a computer. Can you find semialgebraic descriptions of these regions?

4. Let $\mathbb{K}$ be algebraically closed. A subset of $\mathbb{K}^n$ is *constructible* if it is a finite Boolean combination of sets of the form

   $$\mathcal{Z}_{\mathbb{K}}(f) = \{\xi \in \mathbb{K}^n \mid f(\xi) = 0\}$$

   for some $f \in \mathbb{K}[\mathbf{x}] = \mathbb{K}[x_1, \ldots, x_n]$. Describe how you could adapt the proof of Tarski's theorem from class to prove *Chevalley's theorem*: *the projection of a constructible set is constructible.*

5. Let $\mathbb{F} \subset \mathbb{K}$ be two real-closed fields, and consider the categories $\mathbf{SA}_{\mathbb{F}}, \mathbf{SA}_{\mathbb{K}}$ whose objects are semialgebraic sets and morphisms are semialgebraic maps over the appropriate fields. Using Tarski's transfer principle, show there is a covariant, faithful functor $\mathrm{Ext}(\bullet, \mathbb{K}) : \mathbf{SA}_{\mathbb{F}} \to \mathbf{SA}_{\mathbb{K}}$.