

Real AG HW 2 Due October 14.

1. Suppose $\mathbb{K}$ is a real field and $P \subset Q \subset \mathbb{K}$ are positive cones. Prove $P = Q$.
2. If $\mathbb{K}$ is a real field with positive cone P and $\varphi : \mathbb{K} \rightarrow \mathbb{K}$ is any automorphism, prove that $\varphi(P)$ is also a positive cone of $\mathbb{K}$.
3. A *real closure* of an ordered field $(\mathbb{F}, P)$ is another ordered field $(\mathbb{K}, Q)$ where $\mathbb{K}$ is a real-closed algebraic extension of $\mathbb{F}$ and $P = \mathbb{F} \cap Q$. Prove every ordered field has a real closure. (Consider ordered extensions $(\mathbb{K}', Q')$ with $\mathbb{K}'/\mathbb{F}$ algebraic and $\mathbb{K}'$ in some fixed algebraic closure.)
4. Let $A \in \text{Sym}_n(\mathbb{K})$ be a symmetric matrix with entries in a real-closed field $\mathbb{K}$. This exercise outlines a proof of the *spectral theorem* over $\mathbb{K}$.
 - (a) Prove that all eigenvalues of A are real. (Hint: If $\lambda \in \mathbb{K}(i)$ is an eigenvalue with eigenvector $\mathbf{x}$, then $\mathbf{x}^* \mathbf{x} \in \mathbb{K}$, where $\bullet^*$ denotes the conjugate transpose of a matrix with entries in $\mathbb{K}(i)$.)
 - (b) Using the Gram-Schmidt method, construct an $n \times n$ matrix P whose first column is $\mathbf{x} \in \mathbb{K}^n$ (suitably normalized), with $PP^T = \text{id}_n$ and

$$AP = P \begin{pmatrix} \lambda & \mathbf{b}^* \\ \mathbf{0} & B \end{pmatrix}, \quad \mathbf{b} \in \mathbb{K}^{n-1}, B \in \mathbb{K}^{(n-1) \times (n-1)}.$$

- (c) Show $\mathbf{b} = \mathbf{0}$ and $B \in \text{Sym}_{n-1}(\mathbb{K})$. Deduce that A can be written as QDQ^T for $D, Q \in \mathbb{K}^{n \times n}$ with D diagonal and $QQ^T = \text{id}$.
5. This exercise proves *Sylvester's inertia law*. Here, $\mathbb{K}$ denotes a real field, which need not be real-closed. Fix an order on $\mathbb{K}$ and $A \in \text{Sym}_n(\mathbb{K})$.
 - (a) Show that $A = LDL^T$, where $L \in \text{GL}_n(\mathbb{K})$ and D is a diagonal matrix with exactly p positive entries $\lambda_1, \dots, \lambda_p > 0$ and exactly q negative entries $\lambda_{p+1}, \dots, \lambda_{p+q} < 0$ on the diagonal.
 - (b) By part (a), there is an ordered basis $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ for $\mathbb{K}^n$ with

$$\mathbf{v}_i^T A \mathbf{v}_i = \begin{cases} \lambda_i > 0 & \text{if } i \leq p \\ \lambda_i < 0 & \text{if } p < i \leq p+q \\ 0 & \text{else,} \end{cases}$$

and $\mathbf{v}_i^T A \mathbf{v}_j = 0$ for $i \neq j$.

The quadratic form $Q_A(\mathbf{x}) = \mathbf{x}^T A \mathbf{x}$ satisfies

$$Q_A(\mathbf{v}) \leq 0 \quad \forall \mathbf{v} \in \text{span}\{\mathbf{v}_{p+1}, \dots, \mathbf{v}_n\} \subset \mathbb{K}^n.$$

If $W \subset \mathbb{K}^n$ is any subspace such that $Q_A(\mathbf{v}) > 0$ for all $\mathbf{v} \in W \setminus \mathbf{0}$, then W intersects $\text{span}\{\mathbf{v}_{p+1}, \dots, \mathbf{v}_n\}$ trivially. Deduce $\dim(W) \leq p$.

- (c) For another such decomposition $A = L'D'(L')^T$, with p' positive elements and q' negative elements in D' , show $p = p'$ and $q = q'$.