Real AG HW 1 Due September 23

1. Let $P \subset \mathbb{R}^n$ be a polyhedron, $T : \mathbb{R}^n \to \mathbb{R}^m$ any linear map, and let $T|_P : P \to \mathbb{R}^m$ denote the restriction of T to P. Prove $\mathrm{graph}(T|_P) \subset \mathbb{R}^{n+m}$ and $\mathrm{image}(T|_P) \subset \mathbb{R}^m$ are polyhedra.

2. Consider any H-represented polyhedron

$$P = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} \leq \mathbf{b}\}. \tag{1}$$

   (a) Prove that P is closed in the usual Euclidean topology.

   (b) Prove that P is *convex*: this means for any $\mathbf{x}_1, \ldots, \mathbf{x}_k \in P$ that P contains all points of the form

$$\lambda_1\mathbf{x}_1 + \ldots + \lambda_k\mathbf{x}_k, \quad \lambda_1, \ldots, \lambda_k \in [0,1], \quad \lambda_1 + \ldots + \lambda_k = 1. \tag{2}$$

   (c) The *convex hull* of finitely many points $\mathbf{x}_1, \ldots, \mathbf{x}_k \in \mathbb{R}^n$ is the set $\mathrm{conv}(\mathbf{x}_1, \ldots, \mathbf{x}_k) \subset \mathbb{R}^n$ consisting of all convex combinations (2). Prove $\mathrm{conv}(\mathbf{x}_1, \ldots, \mathbf{x}_k)$ is a compact polyhedron (aka a *polytope*.)

   (d) An *extreme point* $\mathbf{x} \in S$ in a convex set $S \subset \mathbb{R}^n$ is any point which cannot be written as a strict convex combination of other points in S: in other words, if $\mathbf{x} = \lambda\mathbf{y} + (1-\lambda)\mathbf{z}$ for $\mathbf{y}, \mathbf{z} \in S$, we must have $\lambda = 0$ or $\lambda = 1$. Prove that $\mathbf{x} \in P$ is an extreme point of the polyhedron (1) if and only if there exists an $n \times (n+1)$ submatrix $(A' \mid \mathbf{b}')$ of $(A \mid \mathbf{b})$ with $\mathrm{rank}\, A' = n$ and $A'\mathbf{x} = \mathbf{b}'$.

   (e) The *Krein-Milman Theorem* from functional analysis implies that any compact, convex subset $S \subset \mathbb{R}^n$ is the convex hull of its extreme points. Using this result, deduce that polytopes are precisely those polyhedra which can be represented as the convex hull of finitely-many points (this is often known as a V-representation.)

3. Consider the simplex $\Delta = \{(x,y) \in \mathbb{R}^2 \mid g_i(x,y) \geq 0,\ i = 1,2,3\}$, where $g_1(x,y) = x$, $g_2(x,y) = y$, $g_3(x,y) = 1 - x - y$. Prove that the polynomial

$$f(x,y) = (3/4)(x^2 + y^2) - xy + 1/2$$

   is positive on Δ by finding a quadratic Handelman representation

$$f = \sum_{\substack{\boldsymbol{\beta} \in \mathbb{Z}^3_{\geq 0} \\ |\boldsymbol{\beta}| \leq 2}} c_{\boldsymbol{\beta}}\, g_1^{\beta_1} g_2^{\beta_2} g_3^{\beta_3}, \qquad c_{\boldsymbol{\beta}} \geq 0.$$

   (This is doable by hand.)

4. Determine the smallest positive integer N such that the polynomial

$$(x+y+z)^N \left((x-y)^2 + y^2 + (x-z)^2\right) \in \mathbb{R}[x,y,z]$$

   has all nonnegative coefficients in the standard monomial basis.
   (For this problem, you probably need a computer.)