Numerical Certification

Numerical Methods ⇝ Proofs

Krawczyk / Moore Criterion: Let V be a FDVS/$\mathbb{R}$, w/ norm $\|\cdot\|$. Let $x \in V$, $r > 0$, $A: V \to V$ linear map.

(In practice: $\|\cdot\|$ is the real ∞-norm on $\mathbb{C}^n$, $\left\| \begin{pmatrix} x_1 + iy_1 \\ \vdots \\ x_n + iy_n \end{pmatrix} \right\| = \max_{i=1}^{n}(|x_i|, |y_i|)$ so B is really a box / interval.)

$f: V \to V$ continuously differentiable. Assume A^{-1} exists.

x s.t. $f(x) \approx 0$, $A \approx df(x)$, want r such that $x + rB$ contains an exact zero: $x_* \in x + rB$ w/ $f(x_*) = 0$.

Assumption $*$: ~~$\forall u, v$~~ $\exists \rho \in (0,1)$ s.t. $\forall u, v \in rB$,

$$-Af(x) + [id_V - A \cdot df(x+u)] \cdot v \in \rho r B.$$

Then there is a unique $x_* \in x + rB$ w/ $f(x_*) = 0$.

(x is called an approximate zero of f w/ associated zero x_*.)

Proof: Define $g(y) = y - A \cdot f(y)$. ~~Assumption $*$ gives~~ $(y = x + u)$

~~$-Af(x) + dg(y)($~~

For $y \in x + rB$, set $u = y - x \in rB$, ~~take~~ so ~~for~~

$$-Af(x) + dg(y) \cdot v \in \rho r B \quad \forall v \in rB$$

Since $B = -B$, also

$$Af(x) + dg(y) \cdot v \in \rho r B \quad \forall v \in rB$$

Thus,

$$\| \pm A \cdot f(x) + dg(y) \cdot v \| \leq \rho r \quad \forall y \in x + rB,\ v \in rB$$

By the triangle inequality,

$$2 \| dg(y) \cdot v \| \leq \| dg(y) - A \cdot f(x) \| + \| dg(y) \cdot v + A \cdot f(x) \|$$

Divide by $2\|v\|$ and maximize over rB:

$$\| dg(y) \|_{op} \leq \rho.$$ This implies $g|_{x+rB}$

is a contractive map: Banach fixed point theorem gives unique $x_* \in x + rB$ such that

$$g(x_*) = x_* \iff x_* = x_* - A \cdot f(x_*)$$
$$\iff f(x_*) = 0.$$

Exercise: using $\|dg(y)\|_{op} < 1$, show that we don't need to assume A^{-1} exists.

Q: How to check Assumption $*$ for all $u, v \in rP$?

A: Use <u>interval</u> <u>arithmetic</u>.

$$\square \mathbb{R} = \{ [a,b] \mid a \leq b,\ a,b \in \mathbb{R} \}$$

$$[a,b] \boxplus [c,d] = [a+c,\ b+d]$$

$$[a,b] \boxtimes [c,d] = [\min(ac, ad, bc, bd),\ \max(ac, ad, bc, bd)]$$

(In practice, w/ floating point, need to round outward.)

$$\square \mathbb{C} = \{ I_1 + i \cdot I_2 \mid I_1, I_2 \in \square \mathbb{R} \}$$

$$\square \mathbb{F}^n = \left\{ \begin{pmatrix} I_1 \\ \vdots \\ I_n \end{pmatrix} \mid I_1, \ldots, I_n \in \square \mathbb{F} \right\} \quad (\mathbb{F} = \mathbb{R} \text{ or } \mathbb{C})$$

For any function $f : \mathbb{C}^n \to \mathbb{C}^m$, an <u>interval extension</u> $\square f : \square \mathbb{C}^n \to \square \mathbb{C}^m$ has the property that

$$\forall\ I \in \square \mathbb{C}^n,\ x \in I,\quad f(x) \in \square f(X).$$

∇ Note: The function $\square f$ is not uniquely determined by f!

For $I \in \square \mathbb{C}^n$, the <u>diameter</u> $\|I\|_\square$ is

$$\|I\|_\square = \max_{x \in I} \|x\| \quad (\text{real } \infty\text{-norm.})$$

To test Assumption (*):

$$\| r^{-1} \cdot A \cdot \square f(x) + (\mathrm{id} - A \cdot \square df(x + rB)) \cdot B \|_{\square} \leq \rho.$$

This is the Krawczyk / Moore test for existence / uniqueness of roots in an interval $x + rB$

Applications: ① Given $f: \mathbb{R}^n \to \mathbb{R}^n$, $x \in \mathbb{C}^n$ w/ $f(x) \approx 0$,

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do we have $x_* \in \mathbb{R}^n$ w/ $x_* \approx x$, $f(x) = 0$?

Consider ~~Set $I_1 = x + rB$, $I_2 = \bar{x}$~~

Check:

① (Krawczyk test)

$$r^{-1} A \,\square f(x) + (\mathrm{id} - A \cdot \square df(x + rB)) \cdot B \subseteq \rho B$$

② If $\|x - \bar{x}\| = \| 2\, \mathrm{Im}(x) \| < \rho \cdot r$, then x & $\bar{x}$ have the same associated zero.

③ If $(x + rB) \cap (\bar{x} + rB) \neq \emptyset$, then the associated zeros of x and $\bar{x}$ are non-real.