MATH 582E Examples Sheet

Tim Duff
University of Washington

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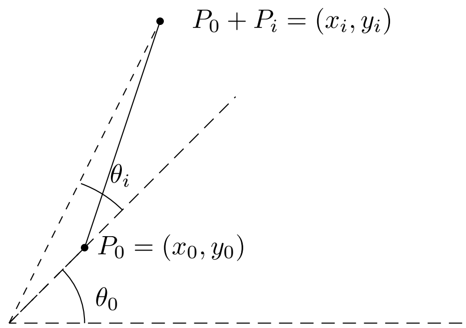

1. You are a robot exploring a flat terrain with a fixed planar coordinate system illustrated above. Your position P_0 and orientation given by the angle θ_0 are unknown. You manage to measure your position and orientation relative to three landmarks P_i whose absolute position and orientation are known in the given coordinate system.

   a Write down a system of 3 polynomial equations in 3 unknowns

   $$X = (x_0, y_0, \tan\theta_0)$$

   which depends polynomially on 9 parameters

   $$P = (x_1, x_2, x_3, y_1, y_2, y_3, \tan\theta_1, \tan\theta_2, \tan\theta_3).$$

   (Hint: write $\tan(\theta_i + \theta_0)$ in two different ways.)

   b) Pick random parameter values; how many solutions are there? How does the number of solutions compare to those predicted by various upper bounds (Bézout, etc.) If applicable, what do any solutions at infinity in $\mathbb{P}^3$ look like?

   c) Consider the polynomial ideal $I \subset \mathbb{K}[X]$ generated by the equations from part 1, where $\mathbb{K} = \mathbb{Q}(P)$ is the function field of the parameter space $\mathbb{A}^9_{\mathbb{Q}}$. Can you compute a Gröbner basis for I?

2. Determine the set of exceptional γ in the complex unit circle such that the homotopy

$$H_\gamma(x;t) = (1-t)f_1(x) + \gamma t f_2(x) = 0$$

has a singular solution $x \in \mathbb{C}$ for some $t \in \mathbb{R}$. For which of the exceptional pairs (γ, t) do we have $t \in [0,1]$?

a) $f_1(x) = x^2 - 1,\ f_2(x) = x^2 + 1$

b) $f_1(x) = x^3 - 2x^2 - x + 2,\ f_2(x) = x^3 + 2x^2 + x + 2$

c) $f_1(x) = x^3 - 2x^2 - x + 2,\ f_2(x) = x^3 - 2x^2 + x - 2$

2. Problems on discriminants:

a) Consider the polynomial

$$p(x,a,b,c) = ax^2 + bx + c$$

and its partial derivative with respect to x

$$\frac{d}{dx}p(x,a,b,c) = 2ax + b.$$

Let I be the ideal generated by p and $\frac{d}{dx}p$ in the in the polynomial ring $R = \mathbb{Q}[x,a,b,c]$: that is,

$$I = \langle ax^2 + bx + c,\, 2ax + b\rangle.$$

Let $<$ be the Lex monomial order such that

$$x > a > b > c$$

What is a Gröbner basis of I with respect to $<$?

b) Apply the Elimination Theorem (Cox Little O'Shea, page 122) to determine a Gröbner basis for the elimination ideal

$$I \cap \mathbb{Q}[a,b,c].$$

You should see a familiar polynomial!

c) Compute the discriminant of $ax^8 + bx^6 + cx^4 + dx^2 + e$ with respect to x. What are its irreducible factors? How does it relate to the discriminant of $ax^4 + bx^3 + cx^2 + dx + e$?

d) Let $f = c_d x^{2d} + c_{d-1}x^{2d-2} + \cdots + c_1 x^2 + c_0$, in which the coefficients $c_d, \ldots, c_0 \in R = \mathbb{K}[y_1, \ldots, y_k]$ are polynomials in some variables other than x. Give necessary and sufficient conditions for when the discriminant of f with respect to x is a square in R.

3. Use Gröbner bases to give an automatic proof of the "power of a point theorem" on page 8 here: `https://arxiv.org/pdf/2105.12058.pdf`.

4. Let $\mathbb{K}$ be any field, and let $<$ be the GrevLex order on $R = \mathbb{K}[x_1, \ldots, x_n]$ satisfying
$$x_1 > x_2 > \cdots > x_n.$$
For homogeneous $f \in R$, prove that f is divisible by x_n iff $\mathrm{in}_<(f)$ is divisible by x_n.

5. For each nonsingular matrix $A \in \mathbb{C}^{4\times 4}$, we may associate a *twisted cubic* which is the image of the map
$$\begin{aligned} \varphi_A : \mathbb{P}^3 &\to \mathbb{P}^3 \\ \begin{pmatrix} s & t \end{pmatrix} &\mapsto \begin{pmatrix} s^3 & s^2t & st^2 & t^3 \end{pmatrix} \cdot A \end{aligned}$$
(Interpret a row vector as the homogeneous coordinates of a point in projective space.) Construct a twisted cubic passing through the points
$$\left(\begin{smallmatrix}1&4&2&3\end{smallmatrix}\right), \left(\begin{smallmatrix}4&2&1&3\end{smallmatrix}\right), \left(\begin{smallmatrix}1&2&3&4\end{smallmatrix}\right), \left(\begin{smallmatrix}2&1&4&3\end{smallmatrix}\right), \left(\begin{smallmatrix}3&2&1&4\end{smallmatrix}\right), \left(\begin{smallmatrix}2&4&3&1\end{smallmatrix}\right).$$
Determine both a unirational parametrization and the ideal of implicit equations.

6. For $n = 2, 3, 4, 5, \ldots$, determine a primary decomposition of the ideal of minors of a $2 \times n$ matrix with distinct indeterminate entries,
$$I_n = \left\langle \det \begin{bmatrix} x_i & x_{i+1} \\ y_i & y_{i+1} \end{bmatrix} \,\middle|\, i = 1, \ldots, n-1 \right\rangle \subset \mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n].$$
Are these ideals radical? Do they have embedded components? How many associated primes will there be for general n?

7. Consider two nested parametric families of bivariate systems:
$$P_1 : \; xy^3 - ay^3 - 1 = x^2 - b = 0$$
$$P_2 : \; xy^3 - ay^3 - 1 = x^2 - a^2 = 0$$
   a) For P_1 and P_2 how many solutions $(x, y) \in \mathbb{C}^2$ are there for generic choices of parameters $(a, b) \in \mathbb{C}^2$? How does this compare to the number predicted by Bézout, Kushnirenko, and Bernstein?
   b) What happens when you use a straight-line homotopy to track solutions to a generic system in P_1 to a generic system in P_2?

8. What do you notice if you use Newton's method to solve
$$f_\epsilon(z) = (z-1)^{10} - \epsilon = 0$$
for $\epsilon = 10^{-7}, 10^{-8}, \ldots$, with, say, $z_0 = 1 + \epsilon^{1/10}/2$ as an initial guess?

9. True or false: $f(z) = z^{10} - 30z^9 + 2$ has a root in the interval
$$[29.999999999999, 29.9999999999999].$$
If true, how many accurate digits can you give for this root?

10. The *Enneper surface* $S \subset \mathbb{R}^3$ is the image of the parametric map

$$\begin{aligned} p_S : \mathbb{R}^2 &\to \mathbb{R}^3 \\ (u, v) &\mapsto (3u + 3uv^2 - u^3, 3v + 3u^2 v - v^3, 3u^2 - 3v^2). \end{aligned}$$

You ask your friend to generate a point $(x, y, z) \in \mathbb{R}^3$ which lies approximately on S. They hand you back the point

$$(x, y, z) = (24275.9, -40229.7, -3658.85).$$

We can be reasonably sure that (x, y, z) does not lie exactly on S, but it may lie "near" S. Consider the following approaches for checking this. Do they give you the same answer? Can you get a certified bound on the distance (say, in the ℓ_2 norm) from (x, y, z) to S?

a) Compute the implicit equation p_S w/ $S = \mathbb{V}(p_S)$, and evaluate $p_S(x, y, z)$.

b) Compute all critical points of the distance function $S \ni (X, Y, Z) \mapsto (X - x)^2 + (Y - y)^2 + (Z - z)^2$. Which is the global maximum?

c) Use pseudowitness sets / homotopy membership test.

11. For any positive integer k, consider a total degree homotopy of the form

$$H(x, y; t) = (1 - t) \begin{bmatrix} x^{2k} - 1 \\ y^{2k} - 1 \end{bmatrix} + \gamma t \begin{bmatrix} * + * x + * x^k y^k \\ * + * y + * x^k y^k \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

for generic choices of coefficients $*$ and randomizing constant γ. As $t \to 1^-$, how many solution curves $((x_i(t), y_i(t))_{i=1\dots 4k^2}$ will diverge?

12. Consider the problem of approximating a given matrix $\widehat{A} \in \mathbb{C}^{n \times n}$ by a matrix of low rank $A \in \mathcal{V}_r$, where $\mathcal{V}_r \subset \mathbb{C}^{n \times n}$ is the smooth, open subvariety of matrices with rank exactly r. The tangent space $T_A \mathcal{V}_r$ to a point $A \in \mathcal{V}_r$ is a complex vector space of dimension $n^2 - (n - r)^2$. We define the *Eckart-Young-Mirsky correspondence* as follows:

$$\mathcal{EYM}_{n,r} = \left\{ (A, \hat{A}) \in \mathcal{V}_r \times \mathbb{C}^{n \times n} \mid \left(A - \widehat{A} \right) \perp T_A \, \mathcal{V}_r \right\}.$$

Coordinate projection onto the second factor

$$\begin{aligned} \pi_A : \mathcal{EYM}_{n,r} &\to \mathbb{C}^{n \times n} \\ (A, \hat{A}) &\mapsto \hat{A} \end{aligned}$$

is a generically finite-to-one map. Points in the fiber $\pi^{-1}(\hat{A})$ may be identified with the critical points of the "squared distance from $\hat{A}$"

$$d(A) = \sum_{1 \le i,j \le n} (a_{i,j} - \widehat{a_{i,j}})^2.$$

What is the monodromy group of π_A?